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"Incompressible flow of a Newtonian fluid past a vertical plate with thermal and magnetic stresses"

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"Incompressible flow of a Newtonian fluid past a vertical plate with thermal and magnetic stresses"

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13. ABSTRACT (Maximum 200 words) I propose to analyze incompressible flow of a Newtonian fluid past a vertical, flat plate with thermal and magnetic stresses. This analysis will include deriving the equations governing the fluid velocity and the temperature distribution. The equations governing fluid velocity will be derived from a force balance approach. We shall consider the forces that act on a differentially small parcel of fluid to determine its behavior. The equations governing temperature will be derived from the principle of conservation of energy. Energy and temperature are closely related. In fact, in an incompressible fluid, temperature is a direct measurement of internal energy. These equations will then be programmed to provide a computer simulation for predicted velocity and temperature fields for various parameters. These simulations will tell us whether or not it is possible to "shape" velocity and temperature distributions using magnetic fields. Possible applications include heat exchanges and any transfer process using fluid flow as a transport medium.

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Abstract

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Abstract

This study is an analysis of incompressible flow of a Newtonian fluid past a vertical, flat plate with thermal and magnetic fields. This analysis will include deriving the equations governing the fluid velocity and the temperature distribution.

The equations governing fluid velocity will be derived from the conservation of linear momentum. Gravitational forces, thermal forces, electromagnetic forces, and viscous forces are considered.

The equations governing temperature will be derived from the principle of conservation of energy. Energy and temperature are closely related. In fact, in an incompressible fluid temperature is a direct measurement of internal energy.

These equations will then be programmed to provide a computer simulation for predicted velocity and temperature fields for various parameters. These simulations will tell us whether or not it is possible to "shape" velocity and temperature distributions using magnetic fields. Possible applications include heat exchangers and any transfer process using fluid flow as a transport medium.

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Chapter One

Definition of Variables and the Geometry of the Study

The first step in formulating a solution to a problem in fluid mechanics is to understand the geometry of the problem and define all variables involved.

This problem involves a Newtonian Fluid in a steady, incompressible flow past a vertical plate with thermal and magnetic fields.

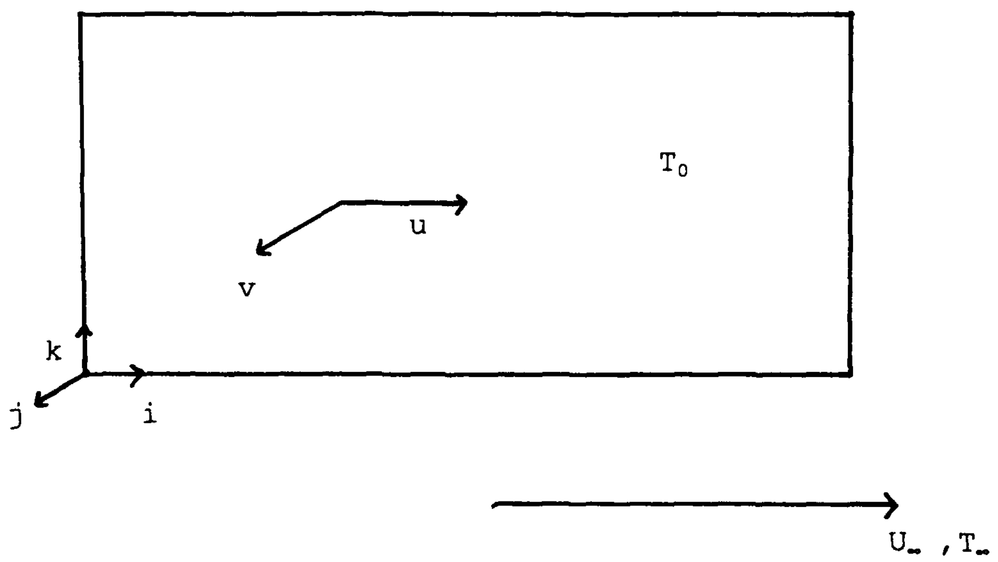

Figure 1-1

Note: this is a left-handed system, therefore i x j = -k.

The following variables will be used throughout the discussion:

$$\vec{v} = \mathbf{v} = velocity = (u\,\hat{\imath} + v\,\hat{\jmath} + w\,\hat{k}),\ \frac{m}{s}$$

$$u = velocity\ in\ the\ x\text{-}direction,\ \frac{m}{s}$$

$$v = velocity\ in\ the\ y\text{-}direction,\ \frac{m}{s}$$

$$w = velocity\ in\ the\ z\text{-}direction,\ \frac{m}{s}$$

$$L = length\ of\ plate,\ m$$

$$T_0 = Temperature\ of\ plate,\ K$$

$$U_\infty = free\ stream\ velocity,\ \frac{m}{s}$$

$$T_\infty = free\ stream\ temperature,\ K$$

$$\mu = absolute\ viscosity = 0.001\ \frac{kg}{m \cdot s}\ for\ water$$

$$\rho = density = 1000\ \frac{kg}{m^3}\ for\ water$$

$$\nu = kinematic\ viscosity = \frac{\mu}{\rho} = 1X10^{-6}\ \frac{m^2}{s}\ for\ water$$

σ = *charge density*, $\frac{C}{m^3}$

$\vec{B} = \mathbf{B}$ = *magnetic field strength*, $\frac{kg}{C \cdot s}$

k = *coefficient of thermal conductivity* , $\frac{kg}{s^3 K}$

C_p = *heat capacity (constant pressure)*, $\frac{J}{kg \cdot K}$

M = *magnetic index number* = $\frac{L\sigma B}{U_\infty \rho}$

Re = *Reynold's number* = $\frac{U_\infty L}{\nu}$

The physical situation being analyzed is an incompressible, Newtonian fluid flowing past a flat plate of length L at a rate U_∞. The plate is at uniform temperature T_0. The ambient fluid is at a different temperature T_∞ . Define the x-direction as the direction along the length of the plate, the y-direction as the direction perpendicular to the plane of the plate, and the z-direction as the vertical direction, perpendicular to both the x- and y-directions. Additionally, define u as the velocity component in the x-direction, v as the velocity component in the y-direction, and w as the velocit component in the z-direction.

The analysis is limited to two dimensions such that the velocity field is a function of x and y and the temperature field is a function of x and y.

Chapter Two

Derivation of Equations Governing Velocity Distribution

In this chapter, the partial differential equations governing fluid motion are derived from the conservation of linear momentum. The result is two equations, one for the balance of forces in the x-direction and one for the balance of forces in the y-direction.

These equations are derived from Newton's Second Law of motion:

$$\sum \vec{F} = \frac{d}{dt}(m\vec{v}) \qquad 2\text{-}1$$

For an incompressible flow, this means:

$$\sum \vec{F} = \rho\, d\forall \frac{d\vec{v}}{dt} \qquad 2\text{-}2$$

Where d∀ is a differential element of volume.
For steady two-dimensional flow Eq. 2-2 can be written as:

$$\sum \vec{F} = \rho\, d\forall \left[\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) \hat{\imath} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right) \hat{\jmath} \right] \qquad 2\text{-}3$$

Two types of forces are considered: body forces and surface forces. Body forces are forces that may be considered to act through the center of mass of the control volume of the fluid. The body forces that are treated are gravity, thermal forces, and electromagnetic forces.

Surface forces act on the surface of a control volume of fluid. Surface forces are expressed as the integral of the stresses over the control surface.[1]

Gravity exerts a force on a differentially-small element of fluid in the following manner:

$$\vec{F}_g = \rho\, d\forall \vec{g} \qquad \textbf{2-4}$$

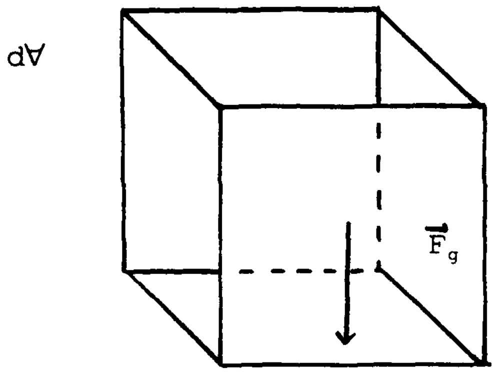

Figure 2-1

Because this force acts only in the z-direction one neglects its effects in horizontal planar flow.

Thermal forces are convective and are caused by changes in fluid density due to temperature[2].

$$\vec{F}_c = \rho\, d\forall \vec{g} \beta (T_1 - T_2) \qquad \textbf{2-5}$$

Since this force acts in the z-direction it may be neglected.

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[1]Robert A. Granger, Fluid Mechanics (New York: Holt, Rhinehart and Winston, 1985) 187.

[2]Adrian Bejan, Convective Heat Transfer (New York: John Wiley and Sons, 1984) 114.

The electromagnetic force is expressed as[3]:

$$\vec{F}_B = q\vec{v} \times \vec{B} = \sigma d\forall \vec{v} \times \vec{B} \qquad \textbf{2-6}$$

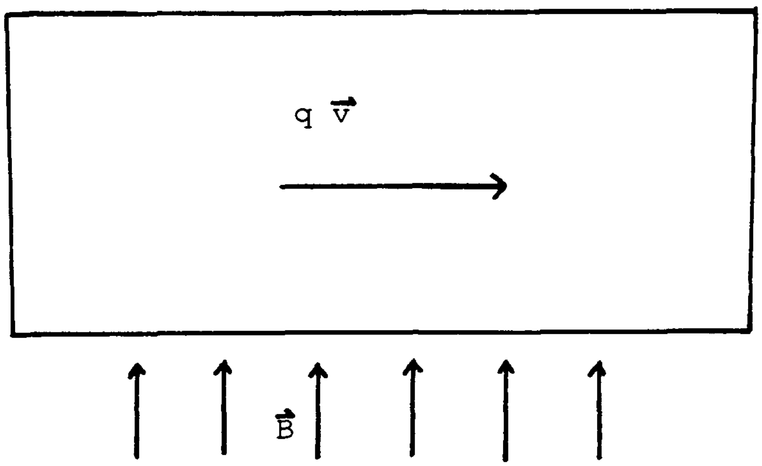

Figure 2-2

Sigma is the charge density and B is the magnetic field strength. Considering only forces in the x- and y-directions Eq. 2-6 simplifies to:

$$\vec{F}_B = \sigma d\forall(-vB_z\hat{i} + uB_z\hat{j}) \qquad \textbf{2-7}$$

Consider next the surface forces. Figure 2-3 shows a differentially-small element of fluid with dimensions (dx,dy,dz), with assorted stresses being applied.[4]

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[3]Paul A. Tipler, Physics for Scientists and Engineers (New York: Worth Publishers Inc., 1991) 783.

[4]Dr. Hermann Schlichting, Boundary Layer Theory (New York: McGraw-Hill Book Co. 1968) 253.

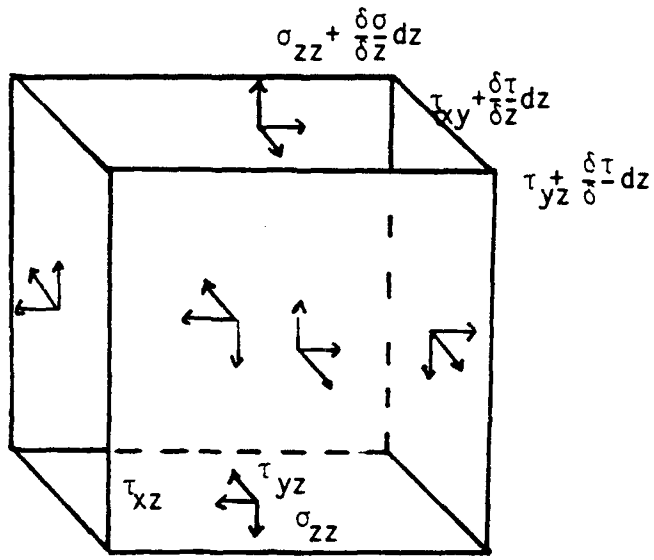

Figure 2-3

Where:

τ_{ab}=*shear stress in the b-direction from shear in the a-plane*

σ = *stress, NOT charge density*

Summing the forces in the x-direction

$$\sum \vec{F}_{s_x} = (\sigma_x + \frac{\partial \sigma_x}{\partial x} dx - \sigma_x)\,\rho\, dydz + (\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z} dz - \tau_{xz})\,\rho\, dxdy$$

$$+ (\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy - \tau_{xy})\,\rho\, dxdz \qquad \text{2-8}$$

By simplifying Eq 2-8 one obtains

$$\sum \vec{F}_{s_x} = (\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z})\,\rho\, dxdydz \qquad \text{2-9}$$

Similarly, the forces in the y-direction simplify to:

$$\sum \vec{F}_{s_y} = \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x}\right) \rho\, dxdydz \qquad \textbf{2-10}$$

From fluid dynamics[5]

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad \textbf{2-11a}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad \textbf{2-11b}$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \textbf{2-11c}$$

$$\sigma_x = -p + 2\mu\frac{\partial u}{\partial x} \qquad \textbf{2-11d}$$

$$\sigma_y = -p + 2\mu\frac{\partial v}{\partial y} \qquad \textbf{2-11e}$$

Substituting Eq 2-11a through 2-11e into Eq 2-9 yields:

$$\sum \vec{F}_{s_x} = \left[-\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right] d\forall \qquad \textbf{2-12}$$

Similarly, the surface forces in the y direction, Eq 2-10, simplify to

$$\sum \vec{F}_{s_y} = \left[-\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right] d\forall \qquad \textbf{2-13}$$

Summing the force components in the x-direction using Eq 2-3, 2-7, and 2-12 results in:

$$\rho\, d\forall\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \left[-\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right] d\forall - \sigma v B_z d\forall \qquad \textbf{2-14}$$

Since the plate is infinitesimally thin, the pressure

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[5]Granger, 184.

is uniform in both the x- and y-directions, hence any derivatives of pressure with respect to x and y are zero. Using this information, Eq 2-14 simplifies to:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{\sigma B_z}{\rho}v \qquad \textbf{2-15}$$

The equation for conservation of linear momentum in the x-direction will be referred to as Eq A. Similarly the conservation of linear momentum in the y-direction becomes:

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\frac{\sigma B_z}{\rho}u \qquad \textbf{2-16}$$

This will be referred to as Eq B.

Chapter Three

Formulation of the Energy Equation

The First Law of Thermodynamics states that the energy accumulated in a control volume is equal to the energy entering the control volume minus the energy leaving the control volume.

The energy entering the control volume (C.V.) and the energy leaving the control volume is composed of[6]:

Rate of energy accumulation in the C.V.= **[1]**

Rate of transfer of energy by fluid flow + **[2]**

Rate of heat transfer by conduction + **[3]**

Rate of internal heat generation - **[4]**

Rate of net work transfer from C.V. to the environment. **[5]**

These terms can be expressed in the following manner:

$$[1]=\Delta x\Delta y\frac{\partial}{\partial t}(\rho e) \qquad 3\text{-}1$$

Since we assume the flow to be steady, the time rate of change of internal energy is zero, so **[1]** = 0.

From figure 3-1,

$$[2]=\rho ve\Delta x-\left[\rho ve+\frac{\partial}{\partial y}(\rho ve)\Delta y\right]\Delta x+\rho ue\Delta y-\left[\rho ue+\frac{\partial}{\partial x}(\rho ue)\Delta x\right]\Delta y \qquad 3\text{-}2$$

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[6]Bejan, 9.

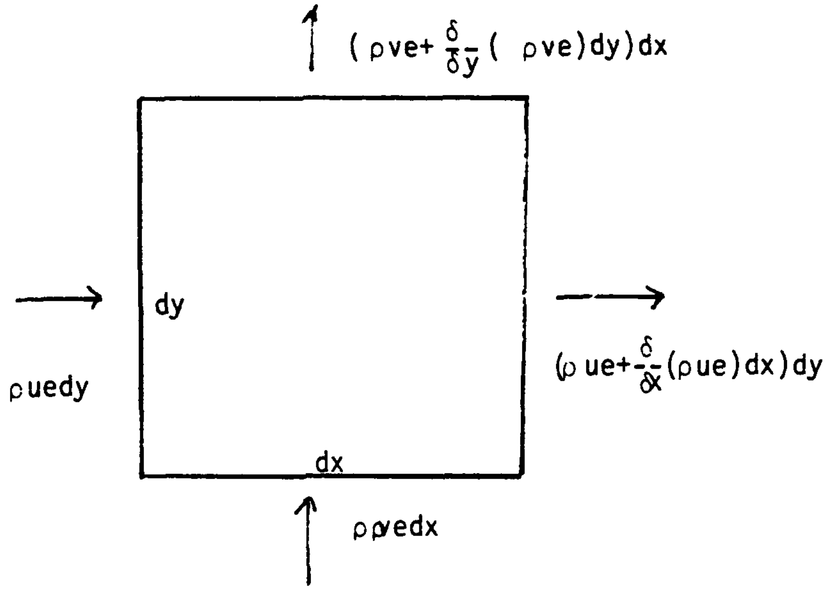


Figure 3-1

Eq 3-2 simplifies to

$$[2] = -\rho \Delta x \Delta y \left(u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y}\right) \qquad \textbf{3-3}$$

From Figure 3-2 one sees that

$$[3] = -(\Delta x \Delta y)\left(\frac{\partial \vec{q}_x}{\partial x} + \frac{\partial \vec{q}_y}{\partial y}\right) = -\Delta x \Delta y \vec{\nabla}\cdot\vec{q} \qquad \textbf{3-4}$$

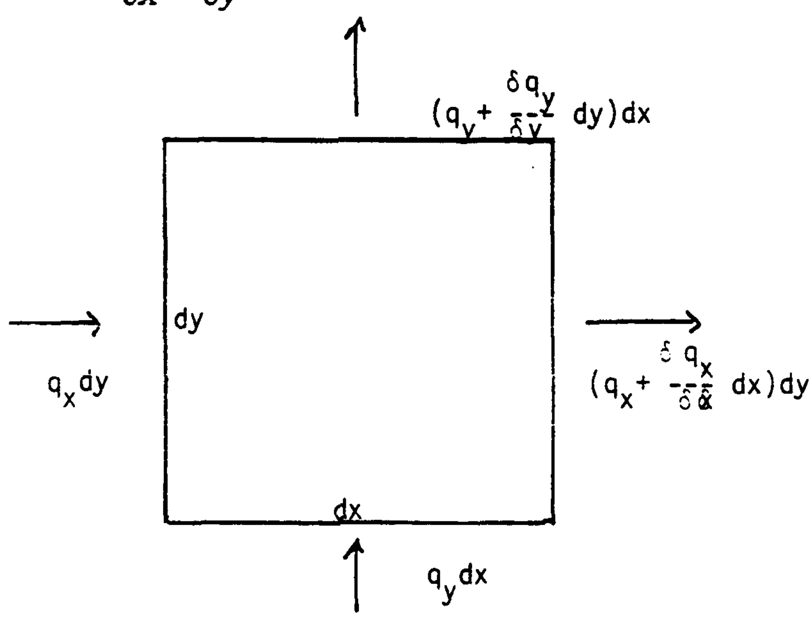


Figure 3-2

The Fourier Law of heat transfer is

$$\ddot{q}=-k\nabla T \tag{3-5}$$

Thus, Eq. 3-4 becomes:

$$\mathbf{[3]}=\Delta x\Delta y k\nabla^2 T \tag{3-6}$$

From Figure 3-2

$$\mathbf{[4]}=\Delta x\Delta y\bar{\ddot{q}} \tag{3-7}$$

No heat is generated in our fluid, so q'''=0 and **[4]** = 0.

In order to calculate **[5]**, the rate of net work transfer from the C.V. to the environment, the forces that exist at the interface between the control volume and the environment must be known. As defined in Chapter Two, these forces are called surface forces. The rate of work transfer is:

$$\mathbf{[5]}=\dot{W}=\frac{d}{dt}\int\vec{F}_S\cdot d\vec{r} \tag{3-8}$$

But for steady flow, the surface forces do not change with respect to time. Thus, Eq 3-8 simplifies to:

$$\mathbf{[5]}=\int\vec{F}_S\cdot\vec{V} \tag{3-9}$$

The above term is analogous to heat produced by fluid friction. The flows that are of concern are slow enough that turbulence may be ignored. Hence the work done by the surface forces will be neglected. In order to determine if this is a realistic assumption, temperature profiles predicted by the resulting energy equation should be compared to experimental results.

Combining all of the above terms results in

[1] = [2] + [3] + [4] - [5]

$$0=-\rho C_p \Delta x \Delta y \left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)+\Delta x \Delta y k \nabla^2 T+0-0 \qquad \textbf{3-10}$$

Rewriting the above yields

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=-\frac{k}{\rho C_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \qquad \textbf{3-11}$$

This equation will be referred to as Eq C.

Chapter Four

Prandtl Order Reduction and Conversion to Dimensionless Variables

The problem has been reduced to three equations in terms of three unknowns: u, v, and T that are functions of space, x and y. The equations may be simplified somewhat by determining if any of the terms are relatively insignificant and may be neglected. The technique of Prandtl Order Reduction can be used to accomplish this.

The first step in the reduction process is to assign each variable an order of magnitude. This indicates the size of the values that the particular variable might represent in comparison to other variables. For example, the velocity u may be assigned an order of magnitude of 1 and the velocity v an order of magnitude of δ, because the expected values of v are much smaller than the expected values of u. Similarly, x has an order of magnitude of 1 and y has an order of magnitude of δ because previous work done in fluid mechanics indicates that velocity is essentially constant except for a small region close to the plate, so one expects the y-dimension to be much smaller than the x-dimension. Recall Eq 2-15

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{\sigma B_z}{\rho}v \qquad \textbf{2-15}$$

Analyzing the orders of magnitude:

$$1\frac{1}{1}+\delta\frac{1}{\delta}=\frac{\mu}{\rho}\left(\frac{1}{1^2}+\frac{1}{\delta^2}\right)-\frac{\sigma B_z}{\rho}\delta \qquad \textbf{4-1}$$

On the left side of the equation, all terms are of order of magnitude 1. The two viscous terms, however, are of different orders of magnitude. One has order of magnitude 1 while the other has an order of magnitude $1/\delta^2$, which is much larger than one. Thus, we neglect the term with order of magnitude 1. Eq 2-15 becomes:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_z}{\rho}v \qquad \textbf{4-2}$$

Using the same method, Eq 2-16 becomes:

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial y^2}\right)+\frac{\sigma B_z}{\rho}u \qquad \textbf{4-3}$$

There now exist three equations relating the variables u, v, T, x, and y. To write these equations in terms of dimensionless parameters, define the following dimensionless qualities:

$$\zeta = \frac{x}{L} \tag{4-4a}$$

$$\eta = \frac{y}{L} \tag{4-4b}$$

$$u^* = \frac{u}{U_\infty} \tag{4-4c}$$

$$v^* = \frac{v}{U_\infty} \tag{4-4d}$$

$$\Theta = \frac{T - T_0}{T_\infty - T_0} \tag{4-4e}$$

Using Eq 4-4a through 4-4e, Eq's 4-2, 4-3, and 3-11 are transformed to dimensionless parameters.

Recall Eq 4-2

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_z}{\rho}v \tag{4-2}$$

$$\left(\frac{L}{U_\infty^2}\right)\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \left(\frac{L}{U_\infty^2}\right)\left(\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_z}{\rho}v\right) \tag{4-5}$$

$$\frac{u}{U_\infty}\frac{\partial \frac{u}{U_\infty}}{\partial \frac{x}{L}} + \frac{v}{U_\infty}\frac{\partial \frac{u}{U_\infty}}{\partial \frac{y}{L}} = \frac{\mu}{\rho U_\infty L}\frac{\partial^2 \frac{u}{U_\infty}}{\partial \left(\frac{y}{L}\right)^2} - \frac{L\sigma B_z}{\rho U_\infty}v \tag{4-6}$$

Eq 4-2 then becomes:

$$u^*\frac{\partial u^*}{\partial \zeta} + v^*\frac{\partial u^*}{\partial \eta} = \frac{1}{Re}\frac{\partial^2 u^*}{\partial \eta^2} - Mv^* \tag{4-7}$$

(Recall our definition of M and Re from Chapter One.)

Similarly, Eq 4-3 becomes:

$$u^*\frac{\partial v^*}{\partial \zeta}+v^*\frac{\partial v^*}{\partial \eta}=\frac{1}{Re}\frac{\partial^2 v^*}{\partial \eta^2}+Mu^* \qquad \textbf{4-8}$$

If we hold T_0 constant

$$\frac{\partial}{\partial x}(T-T_0)=\frac{\partial T}{\partial x}-\frac{\partial T_0}{\partial x}=\frac{\partial T}{\partial x} \qquad \textbf{4-9}$$

The same relationship holds for the derivative of $(T-T_0)$ with respect to y and second derivatives with respect to both x and y. We may use this to convert Eq 3-11 to dimensionless form.

$$\left(\frac{L}{U_\infty}\right)\left(\frac{1}{T_\infty-T_0}\right)\left(u\frac{\partial(T-T_0)}{\partial x}+v\frac{\partial(T-T_0)}{\partial y}\right)=$$

$$\left(\frac{L}{U_\infty}\right)\left(\frac{1}{T_\infty-T_0}\right)\left(-\frac{k}{\rho C_p}\right)\left(\frac{\partial^2(T-T_0)}{\partial x^2}+\frac{\partial^2(T-T_0)}{\partial y^2}\right) \qquad \textbf{4-10}$$

This simplifies to:

$$\left(\frac{u}{U_\infty}\frac{\partial\frac{T-T_0}{T_\infty-T_0}}{\partial\frac{x}{L}}+\frac{v}{U_\infty}\frac{\partial\frac{T-T_0}{T_\infty-T_0}}{\partial\frac{y}{L}}\right)=-\frac{k}{\rho C_p U_\infty L}\left(\frac{\partial^2\frac{T-T_0}{T_\infty-T_0}}{\partial(\frac{x}{L})^2}+\frac{\partial^2\frac{T-T_0}{T_\infty-T_0}}{\partial(\frac{y}{L})^2}\right) \qquad \textbf{4-11}$$

Thus, Eq 3-11 becomes:

$$u^*\frac{\partial\Theta}{\partial\zeta}+v^*\frac{\partial\Theta}{\partial\eta}=-\frac{k}{\rho C_p U_\infty L}\left(\frac{\partial^2\Theta}{\partial\zeta^2}+\frac{\partial^2\Theta}{\partial\eta^2}\right) \qquad \textbf{4-12}$$

Equations 4-12, 4-7, and 4-8 do not fully mathematically state the problem. A set of boundary conditions must be included. The boundary conditions come from physical constraints of the fluid flow.

The first constraint is called the "no slip condition."[7] This states that the velocity of the fluid at the plate is zero.

$$u^{*}=0 \; at \; \eta=0 \qquad \textbf{BC (1)}$$

$$v^{*}=0 \; at \; \eta=0$$

The second constraint is that at distances infinitely far from the plate the velocity approaches the free stream velocity.

$$\vec{v}\rightarrow U_{\infty}\hat{i} \; as \; \eta\rightarrow\infty \qquad \textbf{BC (2)}$$

This leads to

$$u^{*}\rightarrow 1 \; as \; \eta\rightarrow\infty \qquad \textbf{BC (2)}$$

The third boundary condition is at distances infinitely far from the plate the shear stress is zero.

$$\tau=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\rightarrow 0 \; as \; y\rightarrow\infty \qquad \textbf{BC (3)}$$

The technique of Prandtl Order Reduction shows that the partial derivative of v with respect to x may be neglected:

[7] Stuart Churchill, Viscous Flows: the Practical Use of Theory (Boston: Butterworth Publisher, 1988) 256.

$$\tau=\mu\left(\frac{1}{\delta}+\frac{\delta}{1}\right) \qquad \textbf{4-13}$$

The boundary condition of zero shear stress simplifies to

$$\frac{\partial u^*}{\partial \eta}\rightarrow 0 \ as\ \eta\rightarrow 0 \qquad \textbf{BC (3)}$$

Similar conditions exist for the temperature distribution, T. From the definition of the function θ, θ must equal 0 at the plate:

$$\Theta=\frac{T-T_0}{T_\infty-T_0}=\frac{T_0-T_0}{T_\infty-T_0}=0\ at\ \eta=0 \qquad \textbf{BC (4)}$$

Similarly θ must approach 1 at distances infinitely far from the plate:

$$\Theta=\frac{T_\infty-T_0}{T_\infty-T_0}=1\ as\ \eta\rightarrow\infty \qquad \textbf{BC (5)}$$

A final boundary condition exists on the temperature distribution. At distances infinitely far from the plate, the temperature distribution should reach equilibrium and therefore no heat should be transferred. This means that:

$$\frac{\partial\Theta}{\partial\eta}=0\ as\ \eta\rightarrow\infty \qquad \textbf{BC (6)}$$

These boundary conditions, along with the equations, constitute a set of partial differential equations which model the problem.

Chapter Five

Similarity Transformation and First Solution Attempt

Now three equations relating the three unknown functions u(x,y), v(x,y), and T(x,y) in dimensionless form are known and the boundary conditions on the solutions are known. It is possible, however, to express them in fewer variables. Recall that for incompressible fluids:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{5-1}$$

Using this relationship to define a stream function Ψ that relates u and v.

$$u=\frac{\partial \Psi}{\partial y} \rightarrow u^{*}=\frac{u}{U_{\infty}}=\frac{1}{LU_{\infty}}\frac{\partial \Psi}{\partial (\frac{y}{L})}=\frac{1}{LU_{\infty}}\frac{\partial \Psi}{\partial \eta}=\frac{\partial \Psi^{*}}{\partial \eta} \qquad \text{5-2}$$

$$v=-\frac{\partial \Psi}{\partial x} \rightarrow v^{*}=-\frac{\partial \Psi^{*}}{\partial \zeta} \qquad \text{5-3}$$

Using these relationships, Eq 4-7 becomes:

$$\frac{\partial \Psi^{*}}{\partial \eta}\frac{\partial^2 \Psi^{*}}{\partial \eta \partial \zeta}-\frac{\partial \Psi^{*}}{\partial \zeta}\frac{\partial^2 \Psi^{*}}{\partial \eta^2}=\frac{1}{Re}\frac{\partial^3 \Psi^{*}}{\partial \eta^3}+M\frac{\partial \Psi^{*}}{\partial \zeta} \qquad \text{5-4}$$

Similarly, Eq 4-8 becomes:

$$-\frac{\partial \Psi^{*}}{\partial \eta}\frac{\partial^2 \Psi^{*}}{\partial \zeta^2}+\frac{\partial \Psi^{*}}{\partial \zeta}\frac{\partial^2 \Psi^{*}}{\partial \eta \partial \zeta}=\frac{1}{Re}\left(-\frac{\partial^3 \Psi^{*}}{\partial \eta^2 \partial \zeta}\right)+M\frac{\partial \Psi^{*}}{\partial \eta} \qquad \text{5-5}$$

Now the boundary conditions must be described in terms of the stream function. The no slip condition becomes:

$$\frac{\partial \Psi^*}{\partial \eta} = \frac{\partial \Psi^*}{\partial \zeta} = 0 \ at \ \eta = 0 \qquad \textbf{BC (1)}$$

The boundary condition for the streamwise velocity infinitely far from the plate becomes:

$$\frac{\partial \Psi^*}{\partial \eta} \to 1 \ as \ \eta \to \infty \qquad \textbf{BC (2)}$$

The boundary condition on shear stress becomes:

$$\frac{\partial^2 \Psi^*}{\partial \eta^2} \to 0 \ as \ \eta \to \infty \qquad \textbf{BC (3)}$$

For the first attempt at a solution, Ψ^* was assumed to have the form:

$$\Psi^* = A\zeta^a \eta^b f(B\zeta^c \eta^d)$$

The partial derivatives of Ψ^* that appear in Eq 5-4 are:

$$\frac{\partial\Psi^*}{\partial\eta}=Ab\zeta^a\eta^{b-1}f(\epsilon)+ABd\zeta^{a+c}\eta^{b+d-1}f'(\epsilon) \qquad \textbf{5-6a}$$

$$\frac{\partial^2\Psi^*}{\partial\eta^2}=Ab(b-1)\zeta^a\eta^{b-2}f(\epsilon)+ABd(2b+d-1)\zeta^{a+c}\eta^{b+d-1}f'(\epsilon)+$$

$$AB^2d^2\zeta^{a+2c}\eta^{b+2d-2}f''(\epsilon) \qquad \textbf{5-6b}$$

$$\frac{\partial^3\Psi^*}{\partial\eta^3}=Ab(b-1)(b-2)\zeta^a\eta^{b-3}f(\epsilon)+$$

$$ABd[3b(b+2d-2)+(d-2)(d-1)\zeta^{a+c}\eta^{b+d-3}f'(\epsilon)+$$

$$AB^23d^2(b+d-2)\zeta^{a+2c}\eta^{b+2d-3}f''(\epsilon)+AB^3d^3\zeta^{a+3c}\eta^{b+3d-3}f^{(3)}(\epsilon) \qquad \textbf{5-6c}$$

$$\frac{\partial\Psi^*}{\partial\zeta}=Aa\zeta^{a-1}\eta^bf(\epsilon)+ABc\zeta^{a+c-1}\eta^{b+d}f'(\epsilon) \qquad \textbf{5-6d}$$

$$\frac{\partial^2\Psi^*}{\partial\zeta\partial\eta}=Aab\zeta^{a-1}\eta^{b-1}f(\epsilon)+AB(bc+ad+ad)\zeta^{a+c-1}\eta^{b+d-1}f'(\epsilon)$$

$$+AB^2cd\zeta^{a+2c-1}\eta^{b+2d-1}f''(\epsilon) \qquad \textbf{5-6e}$$

Where, for notational purposes, define ε such that:

$$\epsilon=B\zeta^c\eta^d \qquad \textbf{5-7}$$

Substituting Eq 5-6a through 5-6e into Eq 5-4 and simplifying

$$Aab\zeta^{a-1}\eta^{b-2}f^2(\epsilon) +$$

$$AB(bc+ad+bcd-ad^2)\zeta^{a+c-1}\eta^{b+d-2}f(\epsilon)f'(\epsilon) +$$

$$AB^2d(c+ad-bc)\zeta^{a+2c-1}\eta^{b+2d-2}f'(\epsilon)f'(\epsilon) +$$

$$AB^2d(bc+ad)\zeta^{a+2c-1}\eta^{b+2d-2}f(\epsilon)f''(\epsilon)$$

$$-\frac{b(b-1)(b-2)}{Re}\zeta^a\eta^b{-3}f(\epsilon) - AMa\zeta^{a-1}\eta^b f(\epsilon)$$

$$-\frac{Bd[3b(b+2d-2)+(d-1)(d-2)]}{Re}\zeta^{a+c}\eta^{b+d-3}f'(\epsilon) - ABMc\zeta^{a+c-1}\eta^{b+d}f'(\epsilon)$$

$$-\frac{3B^2d(d-1)(d-2)}{Re}\zeta^{a+2c}\eta^{b+2d-3}f''(\epsilon) - \frac{B^3d^3}{Re}\zeta^{a+3c}\eta^{b+3d-3}f^{(3)}(\epsilon) = 0 \quad 5\text{–}8$$

The similarity transformation was chosen to transform the partial differential equation involving the function Ψ into an ordinary differential equation involving the function f. An ordinary differential equation is one where all of the coefficients of the derivatives contain only constants and powers of the argument of the function f. This means that a, b, c, and d must be chosen in such a way that the coefficients of the derivatives of f only involve powers of ϵ, but do NOT contain any free powers of ζ or η. In order to search for values of a, b, c, and d that will satisfy these requirements, a table was created that lists the coefficient of each term and the exponents of ζ and η for that term, as well as the exponents after powers of ϵ were factored out.

Constant	zeta	eta
Aab	a-1	b-2
A(ad+bcd+bc-ad^2)	a+c-1	b+d-2
epsilon	a-1	b-2
Ad(c-bc+ad)	a+2c-1	b+2d-2
epsilon	a+c-1	b+d-2
epsilon^2	a-1	b-2
Ad(bc-ad)	a+2c-1	b+2d-2
epsilon	a+c-1	b+d-2
epsilon^2	a-1	b-2
b(b-1)(b-2)/Re	0	-3
Ma	-1	0
(d/Re)[(d-1)(d-2)+3b(b+d-2)]	c	d-3
epsilon	0	-3
Mc	c-1	d
epsilon	-1	0
3d^2(b+d-1)/Re	2c	2d-3
epsilon	c	d-3
epsilon^2	0	-3
d^3	3c	3d-3
epsilon	2c	2d-3
epsilon^2	c	d-3
epsilon^3	0	-3

Table 3-1

Choose a=0, b=2, c=-1, d=3. Eq 5-8 becomes:

$$-\frac{8A}{B}\epsilon^2 f(\epsilon)\,f'(\epsilon)+\frac{3A}{B}\epsilon^3 f'(\epsilon)\,f'(\epsilon)-\frac{6A}{B}\epsilon^3 f(\epsilon)\,f''(\epsilon)$$

$$--\frac{60}{Re}\epsilon f'(\epsilon)+\frac{M}{B}\epsilon^2 f'(\epsilon)-\frac{108}{Re}\epsilon^2 f''(\epsilon)-\frac{27}{Re}\epsilon^3 f^{(3)}(\epsilon)=0 \qquad \textbf{5-9}$$

Using this definition of Ψ^*, u^* and v^* become:

$$u^{*}=A\eta\,[\epsilon f'(\epsilon)+2f(\epsilon)] \qquad \textbf{5-10}$$

$$v^{*}=AB\zeta^{-2}\eta^{5}f'(\epsilon) \qquad \textbf{5-11}$$

The resulting boundary conditions are:

$$u^{*}=A\eta\,[\epsilon f'(\epsilon)+2f(\epsilon)]=0 \;\; at \;\; \eta=0 \qquad \textbf{BC (1)}$$

$$v^{*}=AB\zeta^{-2}\eta^{5}f'(\epsilon)=0 \;\; at \;\; \eta=0 \qquad \textbf{BC (1)}$$

The no slip conditions are met.

$$u^{*}=A\infty[\infty f'(\infty)+2f(\infty)]=1 \;\; at \;\; \eta=\infty \qquad \textbf{BC (2)}$$

This cannot be used as a boundary condition for the computer solution. The η which appears in Eq 5-10 makes it impossible for u^* to satisfy the boundary conditions. If one defines $f(\infty)=f'(\infty)=0$, one obtains $u^* = 0$ as η approaches infinity, not $u^*=1$ as η approaches infinity, which is a physical constraint of any solution found.

Chapter Six

Second Solution Attempt:

Blasius Analysis

The first similarity transformation attempted did not give a viable solution. The next step is to examine a solution to a similar problem that had been solved many years ago. The solution to the problem of flow past a flat plate without thermal or magnetic fields was solved in 1902 by Blasius using the method of selecting a similarity transformation to change the partial differential equation of the conservation of linear momentum in the x-direction into an ordinary differential equation.[8]

Blasius assumed the following:

$$u=U_\infty F(\epsilon)\,;\quad v=U_\infty G(\epsilon) \qquad \textbf{6-1}$$

where

$$\epsilon=\frac{y}{\delta_0} \qquad \textbf{6-2}$$

and δ was a function of x alone.

The streamline function Ψ becomes

$$\Psi=\int_0^y u\,dy=U_\infty\int_0^\epsilon F(\epsilon)\,\delta_0\,d\epsilon=U_\infty\delta_0 f(\epsilon) \qquad \textbf{6-3}$$

where

[8]Granger, 713.

$$f(\epsilon)=\int_0^{\epsilon} F(\epsilon)\,d\epsilon \; \mathbf{6\text{-}4} \qquad \mathbf{6\text{-}4}$$

This definition yields

$$v=-\frac{\partial\Psi}{\partial x}=-[U_\infty\delta_0' f(\epsilon)+U_\infty\delta_0 f'(\epsilon)\,\frac{d\epsilon}{dx}] \qquad \mathbf{6\text{-}6}$$

$$v^*=-\delta_0' f(\epsilon)+\delta_0'\epsilon f'(\epsilon) \qquad \mathbf{6\text{-}7}$$

$$u=\frac{\partial\Psi}{\partial y}=U_\infty\delta_0 f'(\epsilon)\,\frac{d\epsilon}{dy}\;\rightarrow\; u^*=f'(\epsilon) \qquad \mathbf{6\text{-}8}$$

$$\frac{\partial u^*}{\partial\eta}=\frac{\partial^2\Psi^*}{\partial\eta^2}=\frac{f''(\epsilon)}{\delta_0} \qquad \mathbf{6\text{-}9}$$

$$\frac{\partial u^*}{\partial\zeta}=\frac{\partial^2\Psi^*}{\partial\zeta\partial\eta}=-\frac{\delta_0'}{\delta_0}\epsilon f''(\epsilon) \qquad \mathbf{6\text{-}10}$$

$$\frac{\partial^2 u^*}{\partial\eta^2}=\frac{\partial^3\Psi^*}{\partial\eta^3}=\frac{f'''(\epsilon)}{\delta_0^2} \qquad \mathbf{6\text{-}11}$$

Define:

$$\delta_0'=\frac{d\delta_0}{d\zeta}; \quad f'(\epsilon)=\frac{df(\epsilon)}{d\epsilon} \qquad \mathbf{6\text{-}12}$$

Recall Eq 4-7

$$u^*\frac{\partial u^*}{\partial\zeta}+v^*\frac{\partial u^*}{\partial\eta}=\frac{1}{Re}\frac{\partial^2 u^*}{\partial\eta^2}-Mv^* \qquad \mathbf{4\text{-}7}$$

Substituting 6-6 through 6-11 into Eq 4-7

$$\delta_0\delta_0' f(\epsilon)\,f''(\epsilon)+\frac{1}{Re}f''(\epsilon)-M\delta_0^2\delta_0' f(\epsilon)+M\delta_0^2\delta_0'\epsilon f'(\epsilon)=0 \qquad \mathbf{6\text{-}13}$$

In order to make this an <u>ordinary</u> differential equation, all of the coefficients of the derivatives must involve only constants and powers of ε. For this to be true two conditions must be satisfied at once:

$$\delta_0 \frac{d\delta_0}{d\zeta} = C_1 ; \quad \delta_0^2 \frac{d\delta_0}{d\zeta} = C_2 \qquad \textbf{6-14}$$

The only solution that satisfies this condition is for δ_0 to be a constant. This means that epsilon is a function of y alone, which in turn means that u, which is a function of epsilon, depends on y alone. Experience dictates, however, that u depends also on x.

Chapter Seven

Third Solution Attempt: Modified Blasius Method

The third solution attempt involves slightly modifying the Blasius method. Instead of assuming:

$$u=U_\infty F(\epsilon)\ ;\quad \epsilon=\frac{y}{\delta_0} \tag{7-1}$$

u was assumed to have a slightly different form

$$u=U_\infty x^a y^b F(\epsilon)\ ;\quad \epsilon=\frac{y}{\delta_0} \tag{7-2}$$

Following an analysis similar to that done in the Blasius solution for a flat plate presented in Chapter Six

$$\Psi=\int_0^y u\,dy=\int_0^y U_\infty x^a y^b F\left(\frac{y}{\delta_0}\right)dy=U_\infty x^a\int_0^\epsilon (\epsilon\delta_0)^b F(\epsilon)\,\delta_0 d\epsilon \tag{7-3}$$

$$\Psi=U_\infty x^a \delta_0^{b+1} f(\epsilon)\ ;\quad f(\epsilon)=\int_0^{\epsilon\delta_0}\epsilon^b F(\epsilon)\,d\epsilon \tag{7-4}$$

Using Eq 7-4, one obtains the following expressions:

$$u=\frac{\partial\Psi}{\partial y}=U_\infty x^a\delta_0^b f'(\epsilon) \tag{7-5a}$$

$$v=-\frac{\partial\Psi}{\partial x}=-U_\infty x^{a-1}\delta_0^{b+1}f(\epsilon)-U_\infty(b+1)x^a\delta_0^b\delta_0' f(\epsilon)+U_\infty x^a\delta_0^b\delta_0'\epsilon f'(\epsilon) \tag{7-5b}$$

$$\frac{\partial u}{\partial x}=\frac{\partial^2\Psi}{\partial x\partial y}=U_\infty a x^{a-1}\delta_0^b f'(\epsilon)+U_\infty b x^a\delta_0^{b-1}\delta_0' f'(\epsilon)-U_\infty x^a\delta_0^{b-1}\delta_0'\epsilon f''(\epsilon) \tag{7-5c}$$

$$\frac{\partial u}{\partial y}=\frac{\partial^2 \Psi}{\partial y^2}=U_\infty x^a \delta_0^{b-1} f''(\epsilon) \qquad \textbf{7-5d}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial^3 \Psi}{\partial y^3}=U_\infty x^a \delta_0^{b-1} f'''(\epsilon) \qquad \textbf{7-5e}$$

Entering Eq 7-5a through 7-5e into Eq 4-2 yields the following:

$$U_\infty^2 a x^{2a-1}\delta_0^{2b} f'(\epsilon)\, f'(\epsilon) + U_\infty^2 b x^{2a}\delta_0^{2b-1}\delta_0' f'(\epsilon)\, f'(\epsilon)$$

$$-U_\infty^2 x^{2a}\delta_0^{2b-1}\delta_0'\epsilon f'(\epsilon)\, f''(\epsilon) - U_\infty^2 a x^{2a-1}\delta_0^{2b} f(\epsilon)\, f''(\epsilon)$$

$$+U_\infty^2 x^{2a}\delta_0^{2b-1}\delta_0'\epsilon f'(\epsilon)\, f''(\epsilon) = \frac{\mu}{\rho} U_\infty x^a \delta_0^{b-2} f''(\epsilon) + \frac{B\sigma}{\rho} U_\infty a x^{a-1}\delta_0^{b+1} f(\epsilon)$$

$$+\frac{B\sigma}{\rho} U_\infty (b+1)\, x^a \delta_0^b \delta_0' f(\epsilon) - \frac{B\sigma}{\rho} U_\infty x^a \delta_0^b \delta_0' \epsilon f'(\epsilon) \qquad \textbf{7-6}$$

Simplifying Eq 7-6 by dividing by $U_\infty\ x^a\ \delta_0^{b-2}$ yields:

$$U_\infty [a x^{a-1}\delta_0^{b+2} b x^a \delta_0^{b+1}\delta_0']\, f'f' - U_\infty [a x^{a-1}\delta_0^{b+2} + (b+1)\, x^a \delta_0^{b+1}\delta_0']\, ff''$$

$$-\frac{\mu}{\rho} f''' + \frac{B\sigma}{\rho} a x^{-1}\delta_0^3 f + \frac{B\sigma}{\rho}(b+1)\,\delta_0^2\delta_0' f - \frac{B\sigma}{\rho}\delta_0^2\delta_0'\epsilon f' = 0 \qquad \textbf{7-7}$$

A δ_0 function that makes this an ordinary differential equation must now be found. There are a number of coefficients that involve δ's and powers of x:

1) $x^{a-1}\delta_0^{b+2}=C1$; 2) $x^a\delta_0^{b+1}\delta_0'=C2$; 3) $x^{-1}\delta_0^3=C3$; 4) $\delta_0^2\delta_0'=C4$

Working with condition (3) gives:

$$x^{-1}\delta_0^3 = C3 \rightarrow \delta_0 = C3 x^{\frac{1}{3}} \qquad \textbf{7-8}$$

This is interesting because it indicates that:

$$\epsilon=\frac{y}{x^{\frac{1}{3}}} \rightarrow \epsilon^3=\frac{y^3}{x} \qquad \textbf{7-9}$$

which is similar to the transformation that changed the equation from a partial differential equation into an ordinary differential equation in Chapter Five. In that transformation, a function of y^3 over x was used, whereas in this solution a function of y over $x^{1/3}$ is used.

Next one must check to see if this function of δ_0 satisfies the other conditions. Looking at condition 4:

$$\delta_0^2\delta_0'=(C3x^{\frac{1}{3}})^2(\frac{1}{3}C3x^{-\frac{2}{3}})=\frac{1}{3}C3^3=Constant \qquad \textbf{7-10}$$

Thus condition 4 is satisfied.

Looking at condition 1

$$x^{a-1}\delta_0^{b+2}=x^{a-1}(C3x^{\frac{1}{3}})^{b+2}=Cx^{\frac{3a-3+b+2}{3}}=Constant \rightarrow 3a+b-1=0 \qquad \textbf{7-11}$$

Looking at condition 2

$$x^a\delta_0^{b+1}\delta_0'=x^a(C3x^{\frac{1}{3}})^{b+1}(\frac{1}{3}C3x^{-\frac{2}{3}})=Cx^{\frac{3a+b+1-2}{3}}---$$

$$= Constant \rightarrow 3a+b-1 = 0 \qquad \textbf{7-12}$$

Thus 3a + b - 1 = 0 is a relationship between a and b that must exist in order to satisfy conditions 1 and 2. Now examine the boundary conditions. Recall from Eq 7-2 that this analysis began with the assumption:

$$u=U_{\infty}x^{a}y^{b}F(\epsilon) \rightarrow u^{*}=x^{a}y^{b}F(\epsilon) \qquad \textbf{7-2}$$

One of the boundary conditions is that u approaches 1 as y approaches infinity (BC 2). But as y approaches infinity, u approaches infinity for this definition of u. One way to attempt to solve this dilemma is to modify the boundary condition to state that u^{*} approaches 1 as y approaches some multiple of δ_0 rather than allowing y to approach infinity.

$$u^{*}=x^{a}y^{b}F\left(\frac{y}{x^{\frac{1}{3}}}\right)=x^{a}\left(Cx^{\frac{1}{3}}\right)^{b}F\left(\frac{Cx^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)=c^{b}x^{\frac{3a+b}{3}}F(C)=1 \rightarrow 3a+b=0 \qquad \textbf{7-13}$$

But 3a+b=0 and 3a+b=1 are inconsistent equations. A solution that matches the boundary conditions does not exist.

Note that this is the same situation encountered with the first solution in Chapter Five. In the first solution a transformation was found that yielded an ordinary differential equation, but could not satisfy the boundary conditions. Here, using a very similar transformation, the same problem is encountered.

Chapter Eight

Fourth Solution Attempt:

Constrained Magnetic Field

Recall from the second solution attempt in Chapter Six utilizing the Blasius method of searching for a similarity transformation that Eq 6-13 was:

$$\frac{1}{Re}f^{(3)}(\epsilon)+\delta_0\delta_0'f(\epsilon)f''(\epsilon)-M\delta_0^2\delta_0'f(\epsilon)+M\delta_0^2\delta_0'\epsilon f'(\epsilon)=0 \qquad \text{6-13}$$

Where we had defined the following in Eq 6-12

$$\delta_0'=\frac{d\delta_0}{d\zeta};\ f'(\epsilon)=\frac{df(\epsilon)}{d\epsilon} \qquad \text{6-12}$$

Let M = $M_0\eta^{-1}$. Recall:

$$\epsilon=\frac{\eta}{\delta_0}\rightarrow\delta_0=\frac{\eta}{\epsilon} \qquad \text{8-1}$$

Using Eq 8-1 and the new definition of M to rewrite Eq 6-13:

$$\frac{1}{Re}f^{(3)}(\epsilon)+\delta_0\delta_0'f(\epsilon)f''(\epsilon)-M_0\eta^{-1}\frac{\eta}{\epsilon}\delta_0\delta_0'f(\epsilon)+M_0\eta^{-1}\frac{\eta}{\epsilon}\delta_0\delta_0'\epsilon f'(\epsilon)=0 \qquad \text{8-2}$$

Simplifying Eq 8-2 yields

$$\frac{1}{Re}f^{(3)}(\epsilon)+\delta_0\delta_0'f(\epsilon)f''(\epsilon)-M_0\frac{1}{\epsilon}\delta_0\delta_0'f(\epsilon)+M_0\delta_0\delta_0'f'(\epsilon)=0 \qquad \text{8-3}$$

Now one must change this into an <u>ordinary</u> differential equation. Examining the coefficients of the derivatives in Eq 8-3, one finds that δ_0 δ_0' must be a constant.

$$\delta_0\delta_0'=C \qquad \text{8-4}$$

δ_0 = 0 at x = 0. Thus, Eq 8-4 yields

$$\delta_0=\sqrt{2Cx};\quad \delta_0'=\frac{1}{2}\sqrt{\frac{2C}{x}} \tag{8-5}$$

Verifying Eq 8-5 satisfies Eq 8-4

$$\delta_0\delta_0'=(\sqrt{2Cx})\left(\frac{1}{2}\sqrt{\frac{2C}{x}}\right)=\left(\frac{1}{2}\right)(2C)=C$$

Recall that for the transformation the following expressions were formulated (See chapter Six):

$$u=U_\infty f'(\epsilon) \tag{6-8}$$

$$v=-U_\infty\delta_0' f(\epsilon)+U_\infty\delta_0'\epsilon f'(\epsilon)=(U_\infty)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)[\epsilon f'(\epsilon)-f(\epsilon)] \tag{6-6}$$

Using the boundary conditions on u and v to establish the boundary conditions on the function f(ε).

$$u=U_\infty f'(\epsilon)=0 \;at\; y=0 \rightarrow f'(0)=0 \qquad \textbf{BC (1)}$$

$$v=\left(\frac{1}{2}\right)(U_\infty x^{-\frac{1}{2}})[\epsilon f'(\epsilon)-f(\epsilon)]=0 \;at\; y=0 \rightarrow f(0)=0 \qquad \textbf{BC (1)}$$

$$u=U_\infty f'(\epsilon)=U_\infty \;as\; y\rightarrow\infty,\quad \rightarrow f'(\infty)=1 \qquad \textbf{BC (2)}$$

This constitutes an ordinary differential equation with a set of boundary conditions that may be solved using a computer. Using an approach known as the "shooting" method, Professor Malek-Madani of the United States Naval Academy Math Department wrote a program for Mathematica[tm] software that was capable of solving these equations. The solution method involves treating the problem as an initial value problem and iteratively converging the solution to the underlying boundary value problem. See appendix A for a copy of the Mathematica™ program.

Chapter Nine

Discussion of Results

The computer simulation was used to see what variations in the magnetic field did to the velocity profile. The simulation was run and the outputs are shown for various "Magnetic Index" numbers and Reynolds Numbers. These numbers are a representation of the strength of the magnetic force and the viscous force exerted upon the fluid flow.

Recall from Chapter One that the definition of the Magnetic Index number was:

$$M=\frac{L\sigma B_z}{U_\infty \rho}$$

Some typical figures are

L = 25 cm

σ = 0.001 Molarity (.001 mole free charges / Liter)

B_z = 1.0 Gauss (Approximately twice the strength of the Earth's magnetic Field.)

U = 7 cm/s

ρ = 1000 kg/m^3.

These figures yield a Magnetic Index number M = 0.0344 and a Reynolds Number of Re = 17,500 . This gives an idea of the magnitude of the numbers that should be used in the computer simulation. Appendix B contains computer simulations for various Re and M values.

These graphs include plots of u^* and v^* versus position for various Reynolds numbers and Magnetic Index numbers. Three values of Reynolds number were used: 13000, 17500, and 22000; for each of these Reynolds numbers three values of the Magnetic Index number were used: 0 (to show the velocity profile with no magnetic field), 0.5, and -0.5 . One trial with M = 1.5 was included (Figure B-11) to show exaggerated effects of the magnetic field. For the simulation with Re = 17500 and Mo = 0.5 graphs of $f(\varepsilon)$, $f'(\varepsilon)$ and $f''(\varepsilon)$ versus epsilon were also included.

These graphs are believable because they show a number of characteristics the equations developed from fluid flow theory indicate should appear.

First of all it can be seen that the "boundary layer", the line where the streamwise velocity approaches the free stream velocity, closely follows the boundary layer predicted by the classical Blasius Solution for flow without a magnetic field. This can be seen by comparing any of the simulation outputs with a graph of the boundary layer predicted by classical theory,

$$\delta_{predicted} = 5\sqrt{\frac{\nu x}{U_\infty}} \ .$$

A graph of the predicted boundary layer versus ζ is included after the graph of the first simulation output as Figure B-3. Compare Figure B-1 and Figure B-3 and you can see that for zero magnetic field, this simulation predicts a boundary layer at η equals approximately 0.035 for ζ = 1. Figure

B-3, the predicted boundary layer according to the Blasius Solution, shows a boundary layer of 0.037 for $\zeta = 1$.

A second detail shown in the figures that is supported by theory is that the velocity perpendicular to the plate, v, is affected by changes in M to a larger degree than the streamwise velocity, u. Compare Figure B-1 and B-2 with Figure B-4 and B-5 to see this difference. For the same change in M, the u distribution does not change much (Figure B-1 compared to Figure B-4). A slight hump is produced at the boundary layer in Figure B-4. The v distribution, however, changes much more noticeably (Figure B-2 compared to Figure B-5). In Figure B-5 we see that the v distribution peaks at a value of 0.015 and settles to a value of approximately 0.005, as compared to Figure B-2, where the v distribution peaks at a value of 0.02 and settles there. This difference in the sensitivity to changes in the magnetic field makes sense, because the magnetic force involves the cross product of the magnetic field and the velocity of the charged particles (See Eq 2-6). The velocity in the streamwise direction (x-direction) is much larger than the velocity perpendicular to the plate (y-direction). This means that the magnetic force operating in the y-direction is greater than the magnetic force operating in the x-direction, so a change in M should have a greater affect on the velocity in the y-direction.

Along the same lines, we notice that the "hump" produced by M = 0.5 is slightly larger for larger Reynolds Numbers. This can be seen by looking at Figure B-15 and Figure B-21. Figure B-15 corresponds to a slower flow than Figure B-21, and the "hump" caused by the magnetic field is less noticeable for this slower flow. This corresponds to theory because Reynolds Number is directly proportional to the free stream velocity. A higher Reynolds number equates to a higher velocity, which in turn equates to larger magnetic forces.

These figures show some interesting trends. One observation that can be made is that the slope of the velocity fields, both u and v, is affected by changing M. By comparing Figures B-1 and B-4, one can see that as M increases from 0 to 0.5, u approaches the free stream velocity more quickly. In fact, u appears to overshoot the free stream velocity and then settle back to match the free stream velocity. This overshoot is even more exaggerated in Figure B-11, where u reaches a value of 1.2 times the free stream velocity then settles to a final value matching the free stream velocity. Conversely, u appears to "flatten out" when M changes from 0 to -0.5, as can be seen by comparing Figures B-1 and B-9. The velocity u reaches a higher final value in Figure B-9 than in Figure B-1, but approaches this final velocity more slowly.

The effect the of magnetic field on the slope is even more evident in the graphs of the v distributions.

Comparing Figures B-2 and B-5 show that as the magnetic field increases from 0 to 0.5, v changes the same way that u changed. Figure B-5 shows that the v distribution rises more sharply than in figure B-2, then settles. Similarly, Figure B-10 shows that for a negative M, the v distribution rises more slowly but settles at a higher value than the v distribution in Figure B-2. This indicates that magnetic fields can be used to distribute flows more evenly or create sharper boundary layers.

The figures also show that negative values of M result in higher velocities at greater distances from the plate. This can be seen by comparing figure B-1 and B-9, B-13 and B-17, and B-19 and B-23. In each case the velocities u and v reach higher values for negative values of M. It may be possible to use magnetic fields to control the speed of flows.

What these simulations indicate is that the velocity profile <u>can</u> be shaped by using a magnetic field. This could have many applications. Fluid flow is used mostly as a process of transporting something, whether it is heat being transported by a pipe in a radiator or oxygen being transported by blood in an artery, and these processes of transportation often depend on fluid velocity. Thus, controlling the shape of the velocity profile, allows control of the transport process. One could transport heat through a pipe with less heat loss if one could shape the velocity profile properly. One could aid in the diffusion of oxygen across artery membranes by controlling the boundary layer of blood flow.

The next step that needs to be taken in this research is to validate the computer simulations by running trials and taking measurements to verify the predicted results with experimental data. This study was supposed to generate empirical data, but failed to do so. This was due in part to the fact that the laser doppler velocimeter was inoperable. A hot film anemometer was tried, but this caused a number of problems.

In addition to this, the computer simulation must be expanded to include Eq (C), the Energy Conservation Equation, so a graph of the predicted temperature profile can be created in the same way that graphs of u and v were created.

Bibliography

Bejan, Adrian. Convective Heat Transfer. New York: John Wiley and Sons, 1984.

Bluman, G.W. and J.D. Cole. Similarity Methods for Differential Equations. New York: Springer-Verlag, 1974.

Churchill, Stuart Winston. Viscous Flows: the Practical Use of Theory. Boston: Butterworths, 1988.

Granger, Robert A. Fluid Dynamics. New York: Holt, Rhinehart and Winston, 1985.

Schlichting, Hermann. Boundary-Layer Theory. New York: McGraw Hill, 1968.

Welty, James, Charles Wicks and Robert Wilson. Fundamentals of Momentum, Heat, and Mass Transfer. New York: John Wiley and Sons, 1976.

Zauderer, Erich. Partial Differential Equations of Applied Mathematics. New York: John Wiley and Sons, 1989.

Appendix A:

Mathematica Program

```
f[x1_, x2_, x3_, t_] = x2; g[x1_, x2_, x3_, t_] = x3;
re = 17500; M0 = .5;
C1 = re M0/2;
C2 = re/2;
h[x1_, x2_, x3_, t_] = C1 t^(-1) x1 - C1 x2 - C2 x1 x3;

(*We define f=f(epsilon), g=f'(epsilon), and h = f''(epsilon)*)
(*Then we solve our equation for f'''(epsilon) = h'.*)

tstart = 0.000001;
tfinal = 6;

F[b_] := Module[{sol} , sol = NDSolve[{x1'[t] ==  f[x1[t], x2[t], x3[t],t],
                x2'[t] == g[x1[t], x2[t], x3[t],t],
                x3'[t] == h[x1[t], x2[t], x3[t],t],
                x1[tstart] == 0, x2[tstart] == 0, x3[tstart] == b}, {x1, x2, x3},
                {t, tstart, tfinal}];
                out = Evaluate[{x1[t], x2[t], x3[t]} /. sol/. t -> tfinal];
                {out[[1,2]]- 1}]

(*This module takes has the initial value of f''(epsilon) as a variable*)
(*and gives the final value of f'(epsilon) - 1 as output*)

e = FindRoot[F[b][[1]], {b, 1, 0.9}];

(*This tells us which initial value of f'' minimizes *)
(*the value of f'(tfinal) - 1 *)

soll = NDSolve[{x1'[t] ==  f[x1[t], x2[t], x3[t],t],
                x2'[t] == g[x1[t], x2[t], x3[t],t],
                x3'[t] == h[x1[t], x2[t], x3[t],t],
                x1[tstart] == 0, x2[tstart] == 0, x3[tstart] == e[[1,2]]},
                {x1, x2, x3}, {t, tstart, tfinal}];

(*This solves the system of equations we have set up using*)
(*the initial value for f'' we calculated using the FindRoot function.*)

plot1 = Plot[x1[t] /. soll, {t, tstart, tfinal},
            AxesLabel -> {"epsilon","f(epsilon)"},
            PlotLabel->{" = Re" re," = Mo, graph of f(z)"M0}]

PSPrint[plot1]

plot2 = Plot[x2[t] /. soll, {t, tstart, tfinal},
            AxesLabel -> {"epsilon","f'(epsilon)"},
            PlotLabel->{" = Re" re," = Mo, graph of f'(z)"M0}]

PSPrint[plot2]

plot3 = Plot[x3[t] /. soll, {t, tstart, tfinal},
            AxesLabel -> {"epsilon","f''(epsilon)"},
            PlotLabel->{" = Re" re," = Mo, graph of f''(z)"M0}]

PSPrint[plot3]
```

```
plot3 = Plot[x3[t] /. sol1, {t, tstart, tfinal},
            AxesLabel -> {"epsilon","f''(epsilon)"},
            PlotLabel->{" = Re" re," = Mo, graph of f''(z)"MO}]

PSPrint[plot3]

u[t_] = x2[t] /. sol1;

v[t_] = .5 (t x2[t] - x1[t]) /. sol1;

plot4 = Plot3D[u[y/Sqrt[x]][[1]], {x, 0.1, 1}, {y, 0.00001, .4},
              PlotLabel -> {" = Re" re, " = Mo" MO,"graph of u(x,y)"},
              AxesLabel -> {"zeta","eta","u(x,y)"},
              PlotPoints-> 30]

plot6 = Plot3D[x^(-.5) v[x^(-.5) y][[1]], {x, 0.1, 1}, {y, 0.00001, .4},
              PlotLabel -> "graph of v(x,y)",
              AxesLabel -> {"zeta","eta","v(x,y)"},
              PlotPoints-> 30]
```
□

Appendix B:

Computer

Simulation Trials

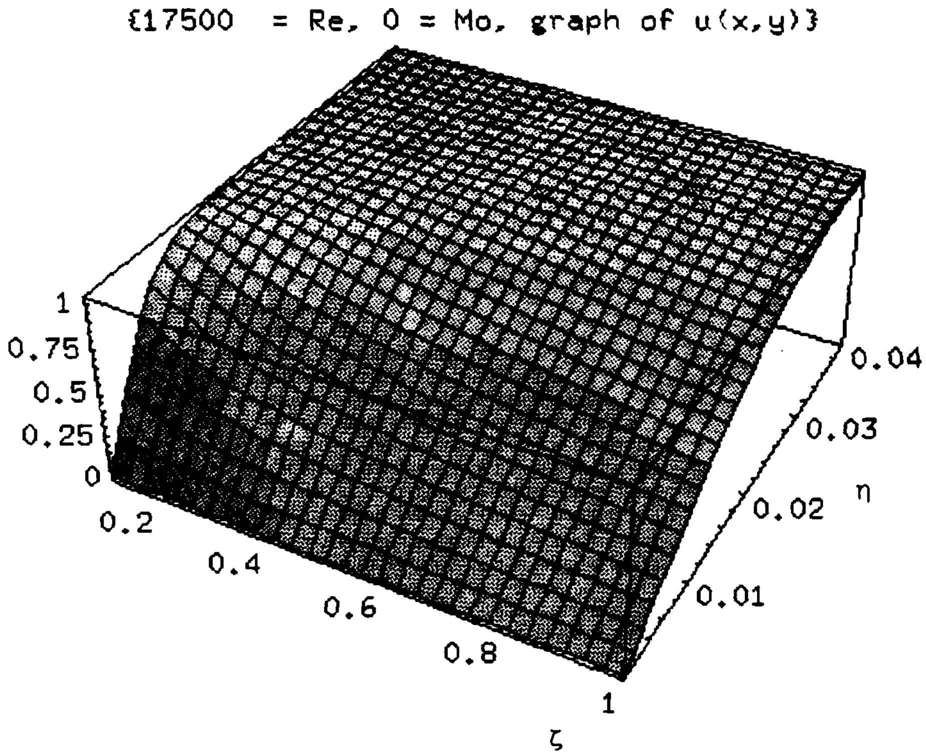


Figure B-1

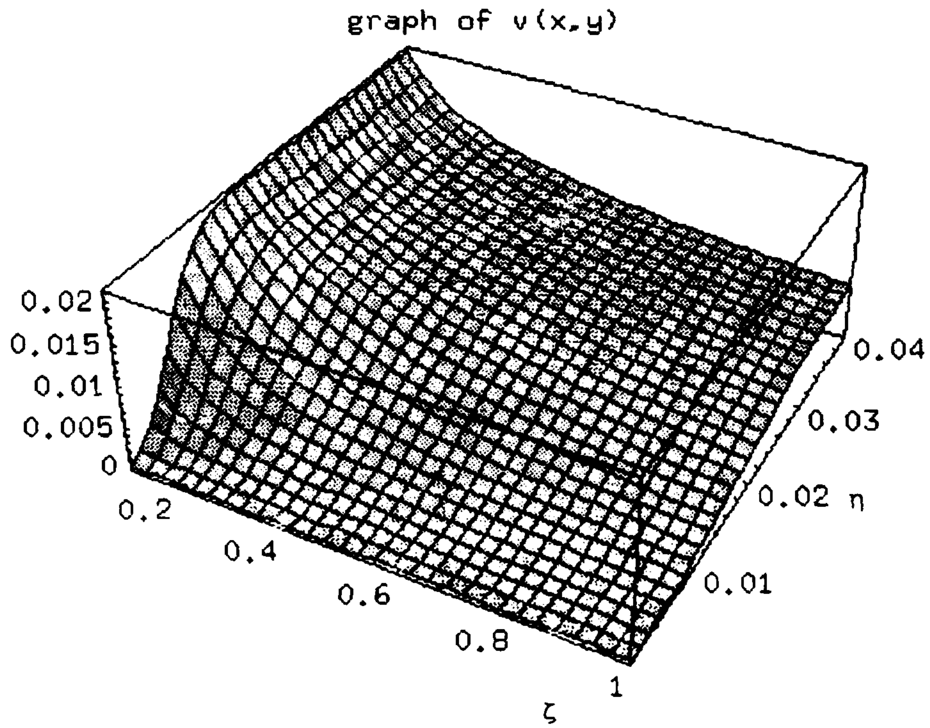


Figure B-2

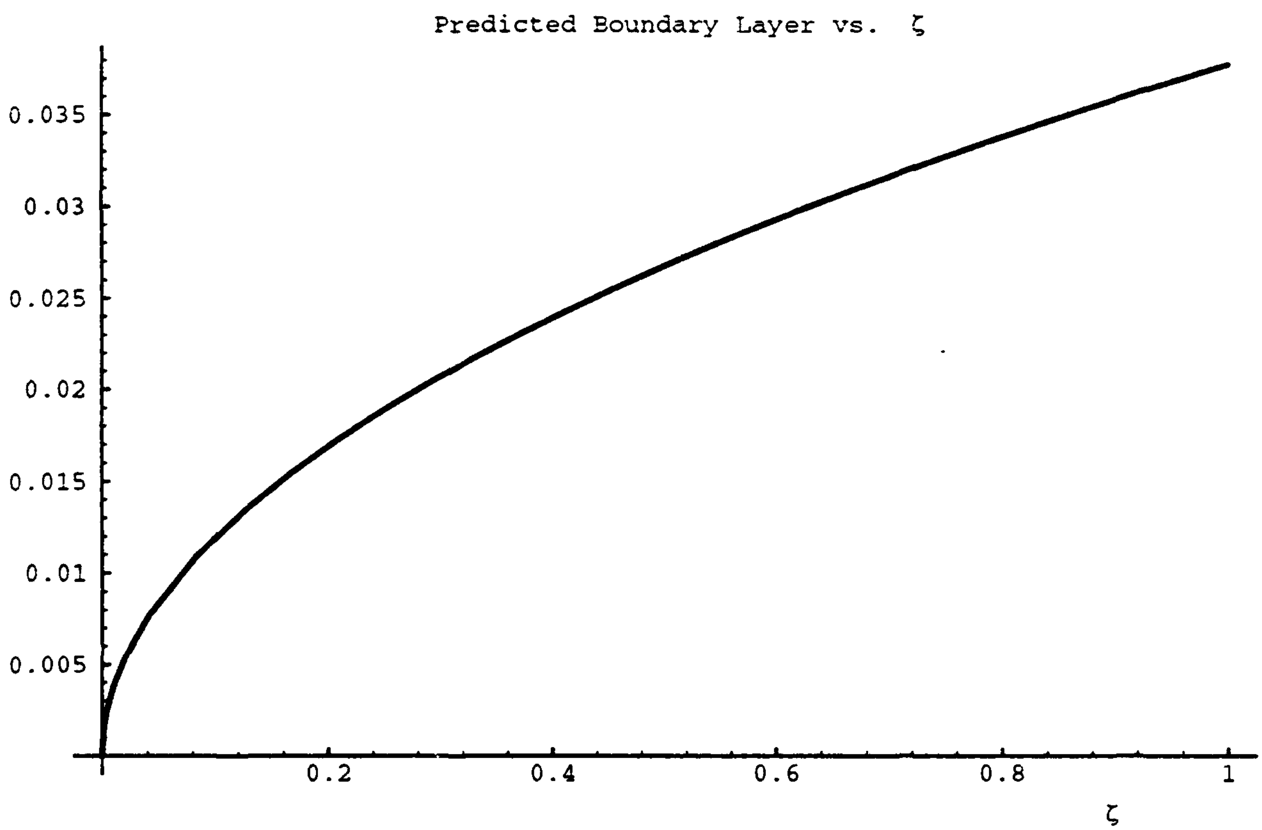

Predicted Boundary Layer vs. ζ
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0.2
0.4
0.6
0.8
1
ζ

Figure B-3

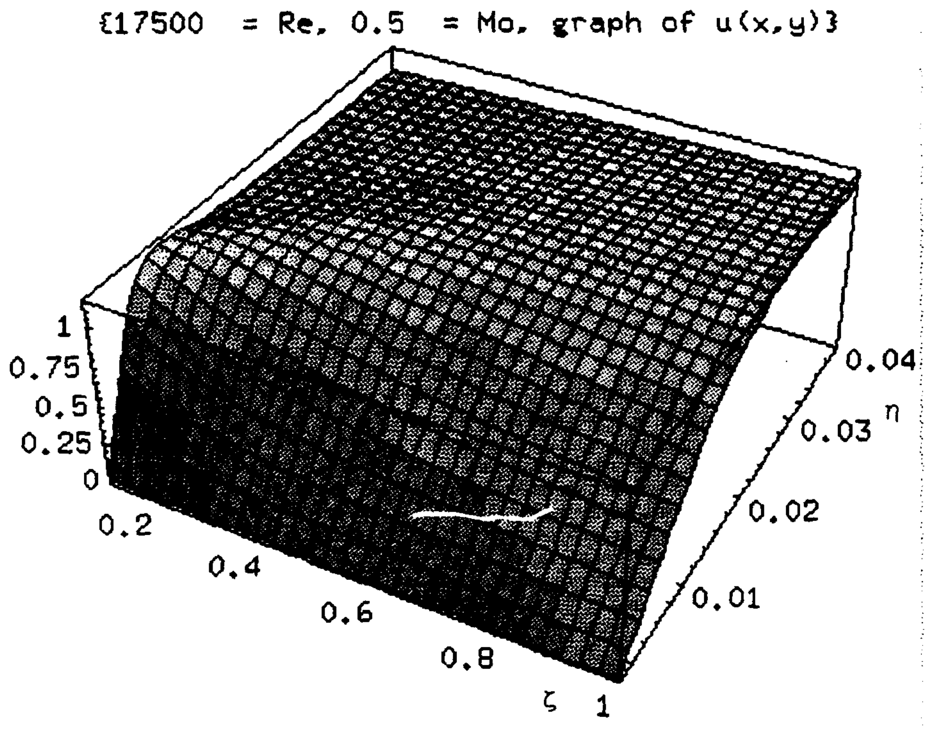


Figure B-4

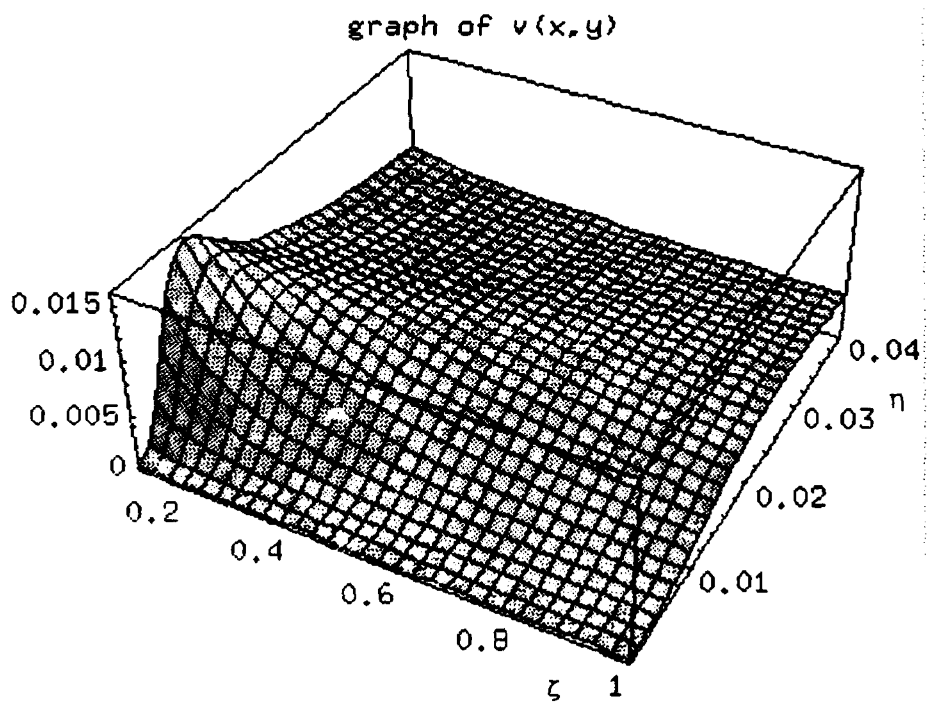


Figure B-5

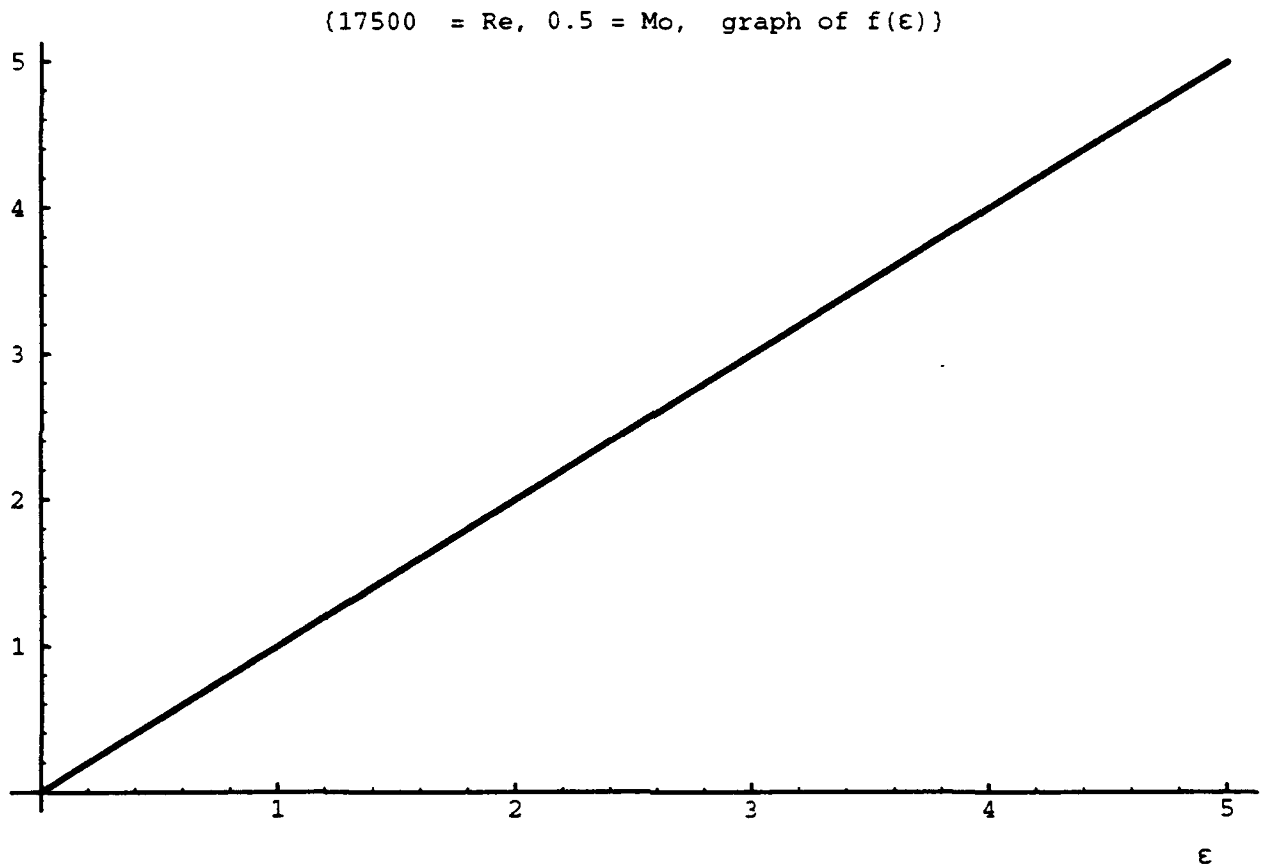


Figure B-6

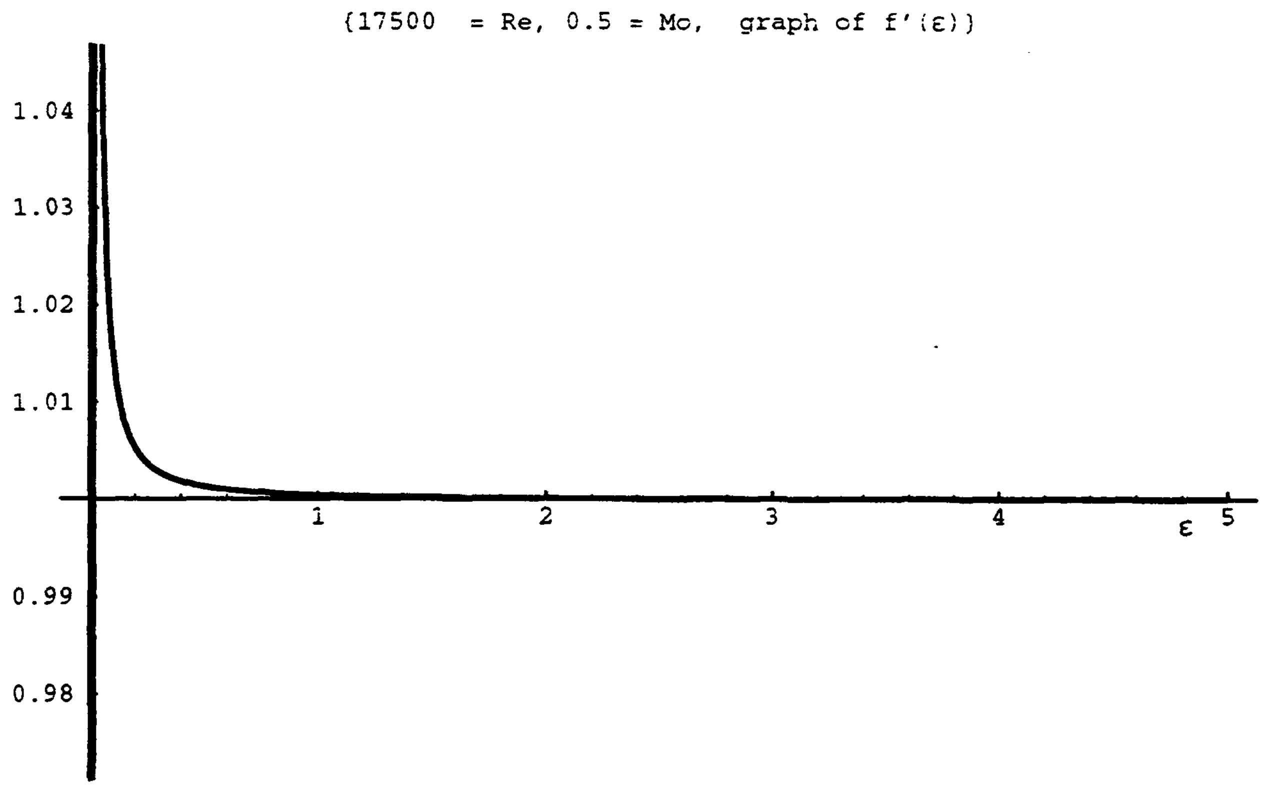


Figure B-7

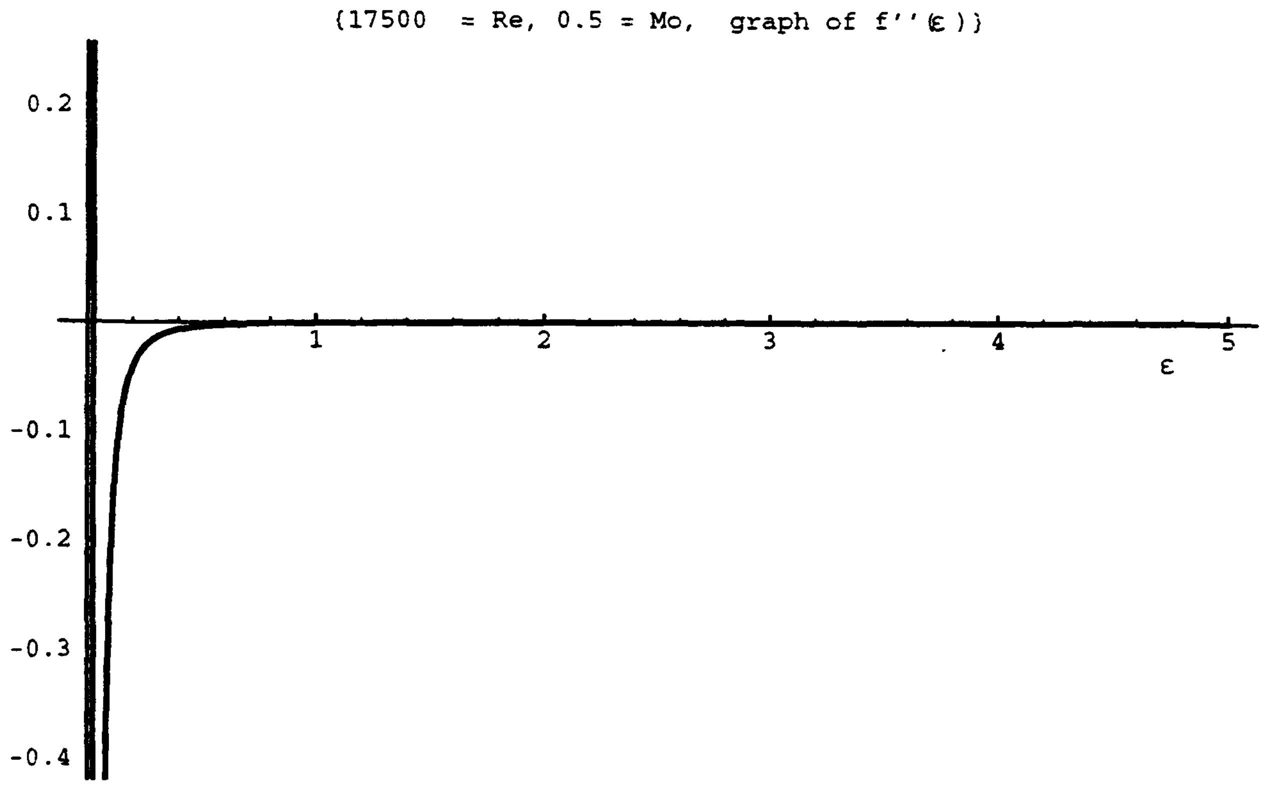


Figure B-8

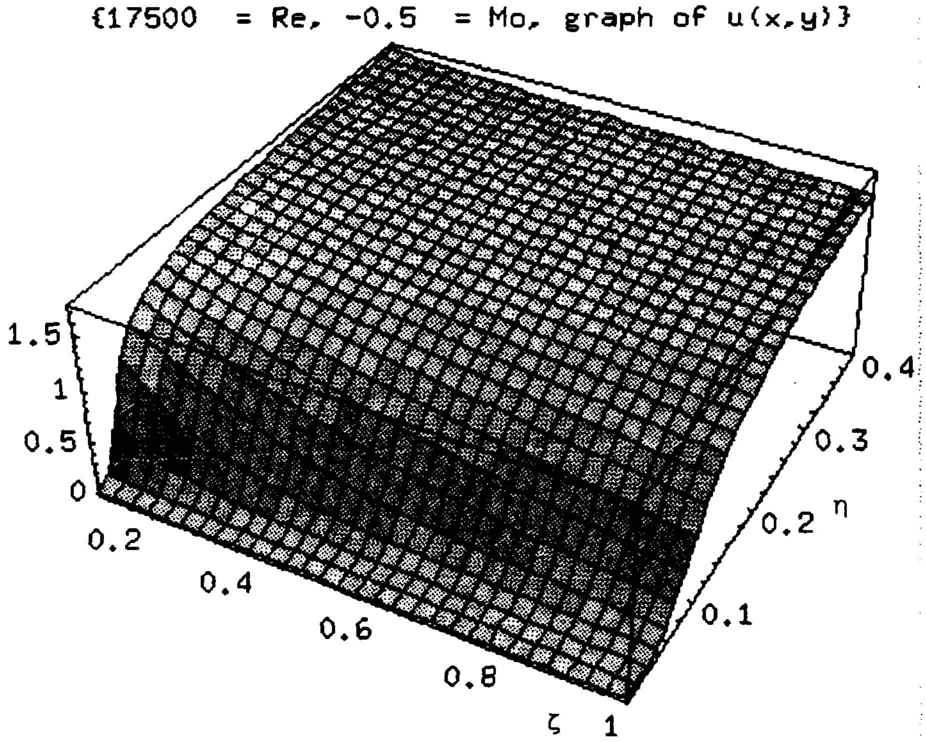


Figure B-9

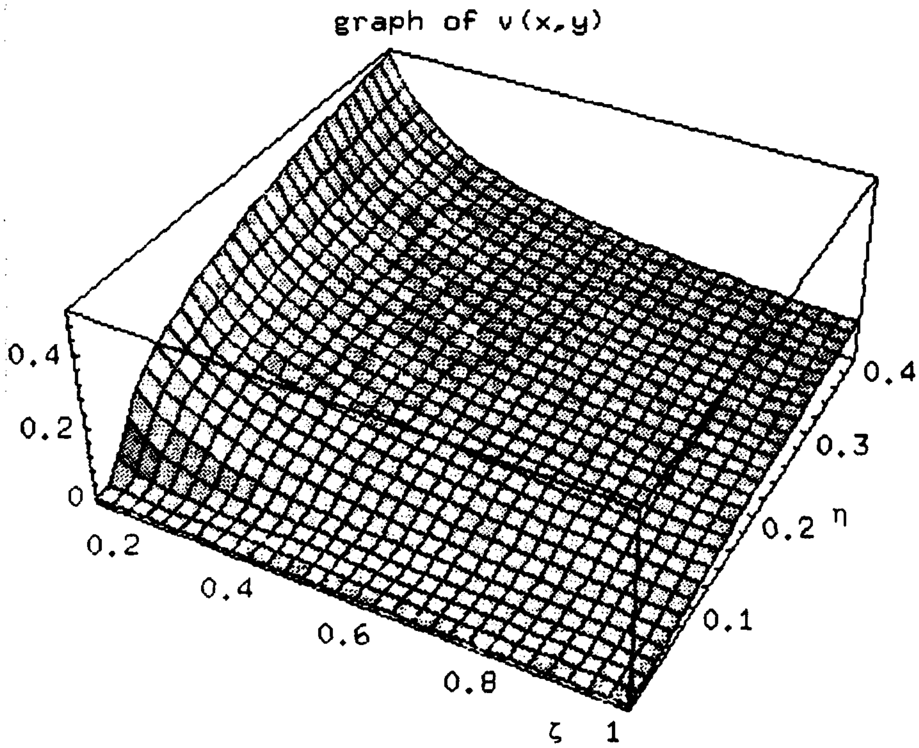


Figure B-10

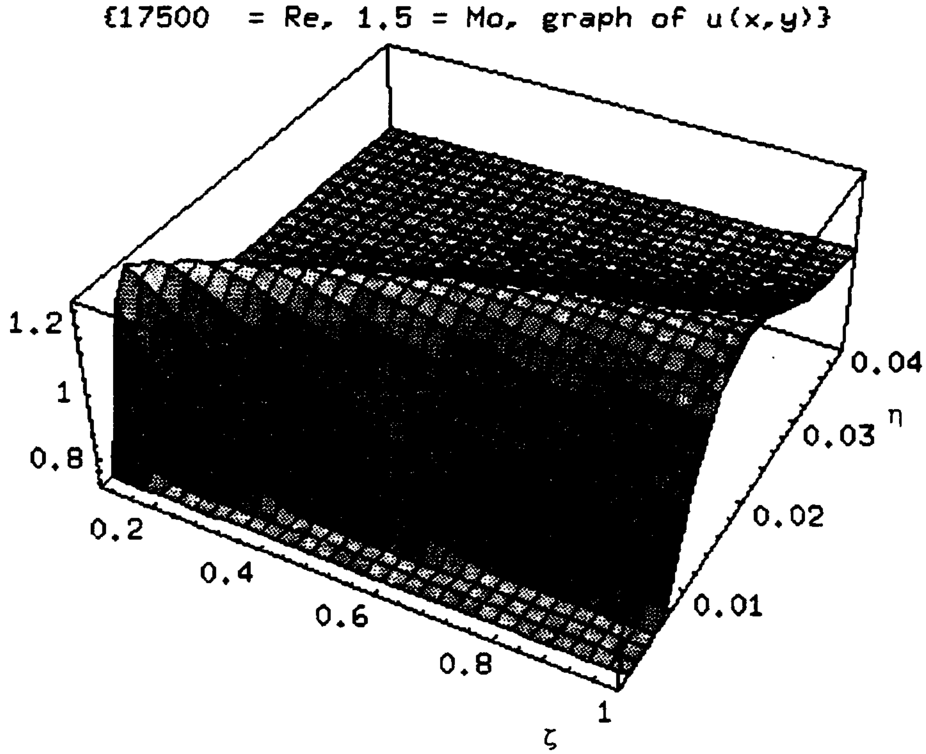
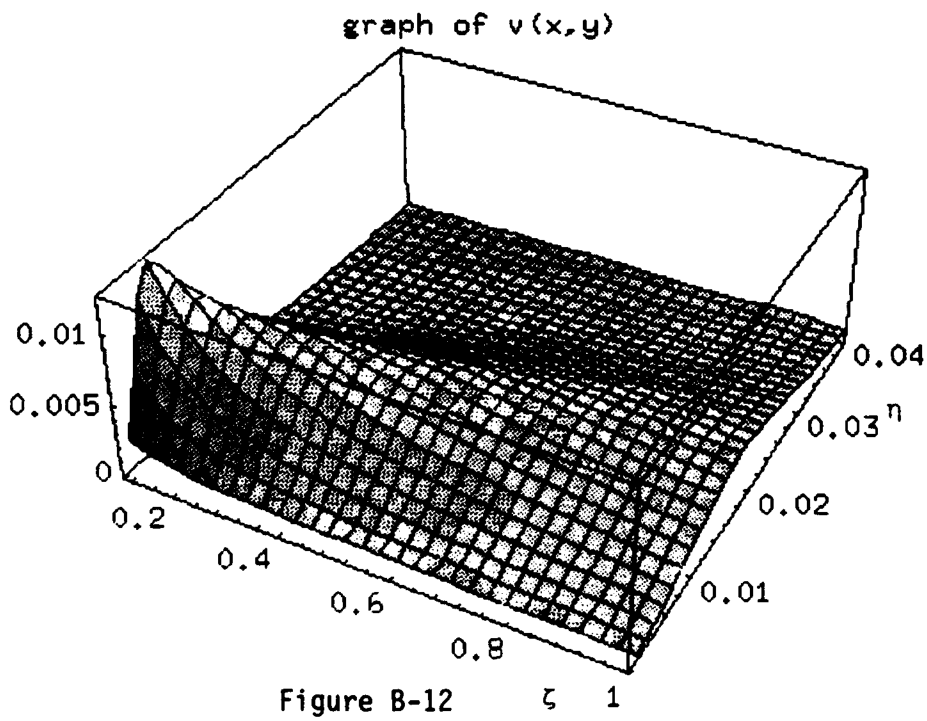


Figure B-11



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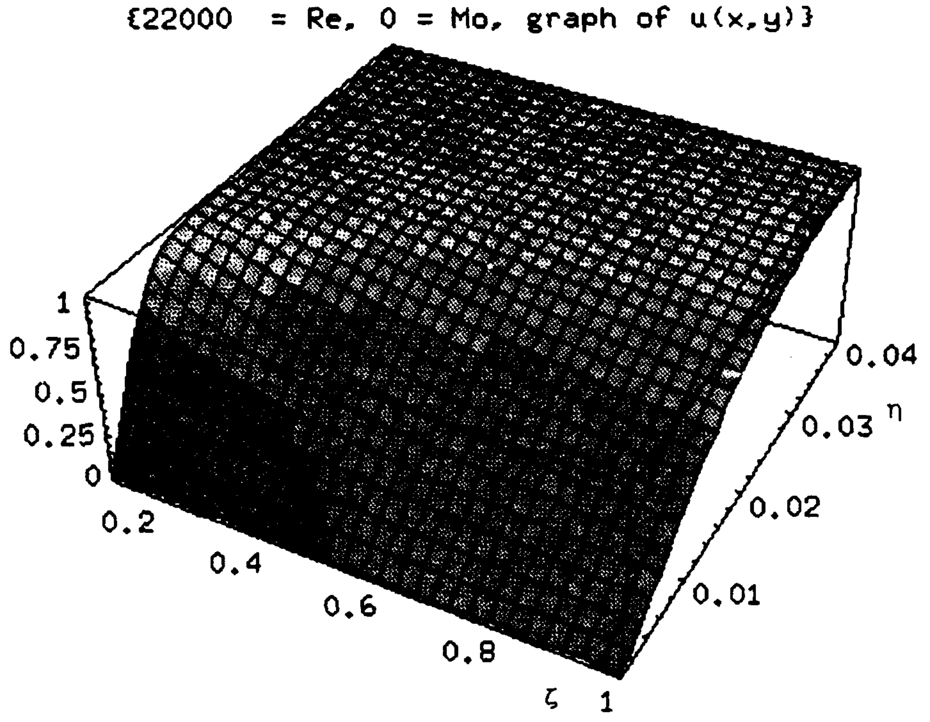


Figure B-13

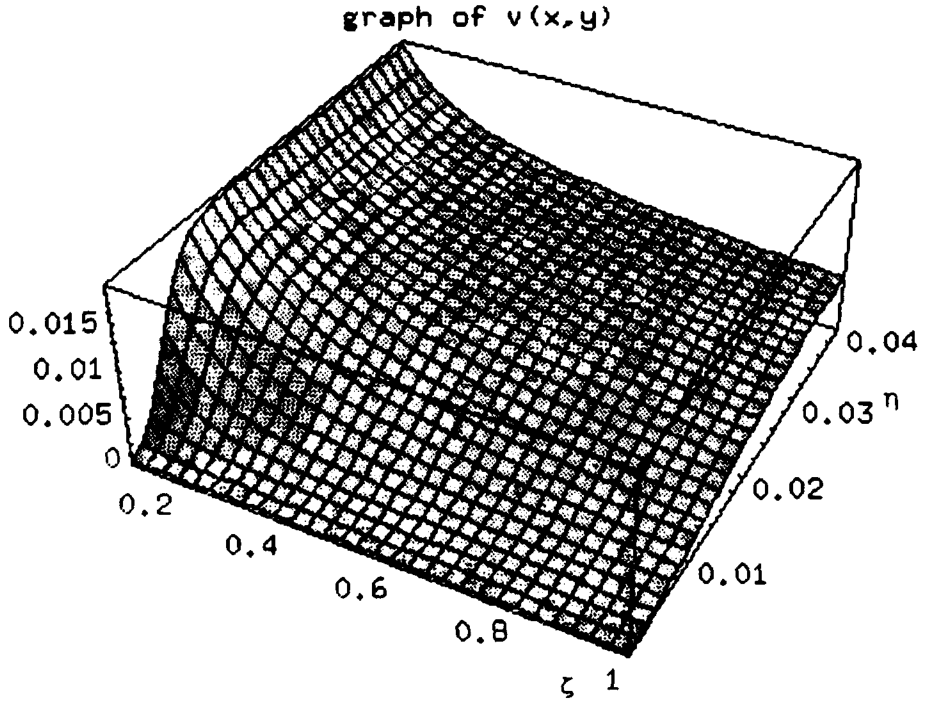
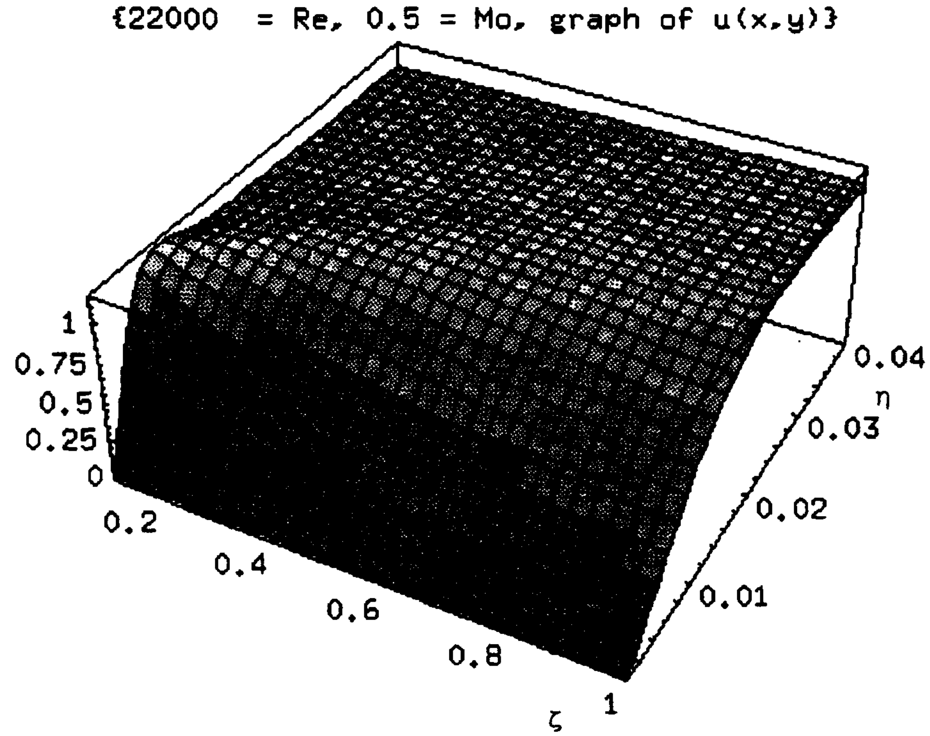


Figure B-14



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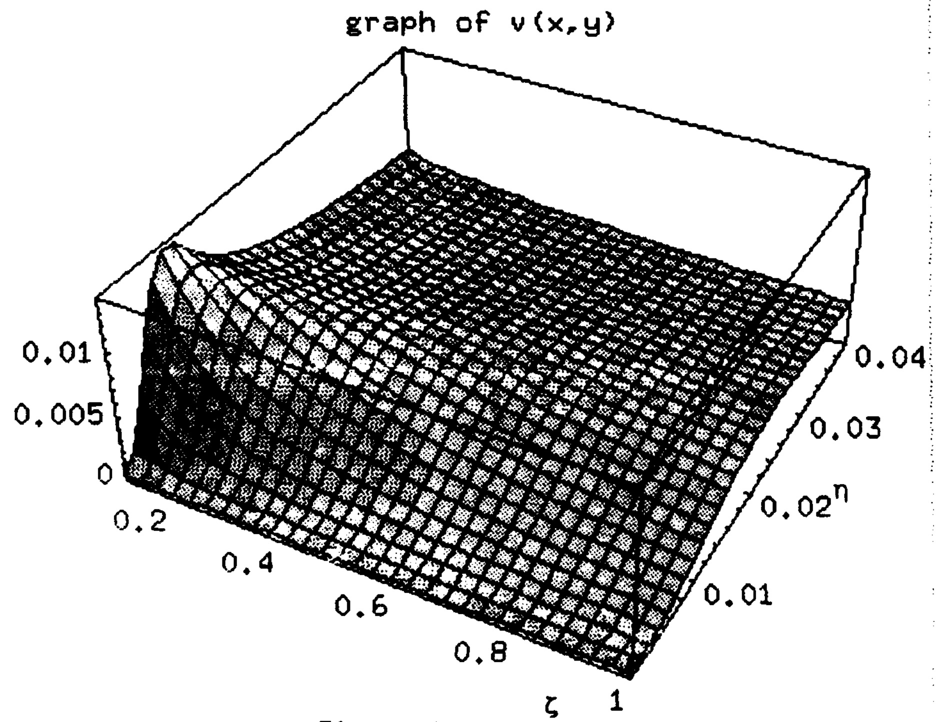
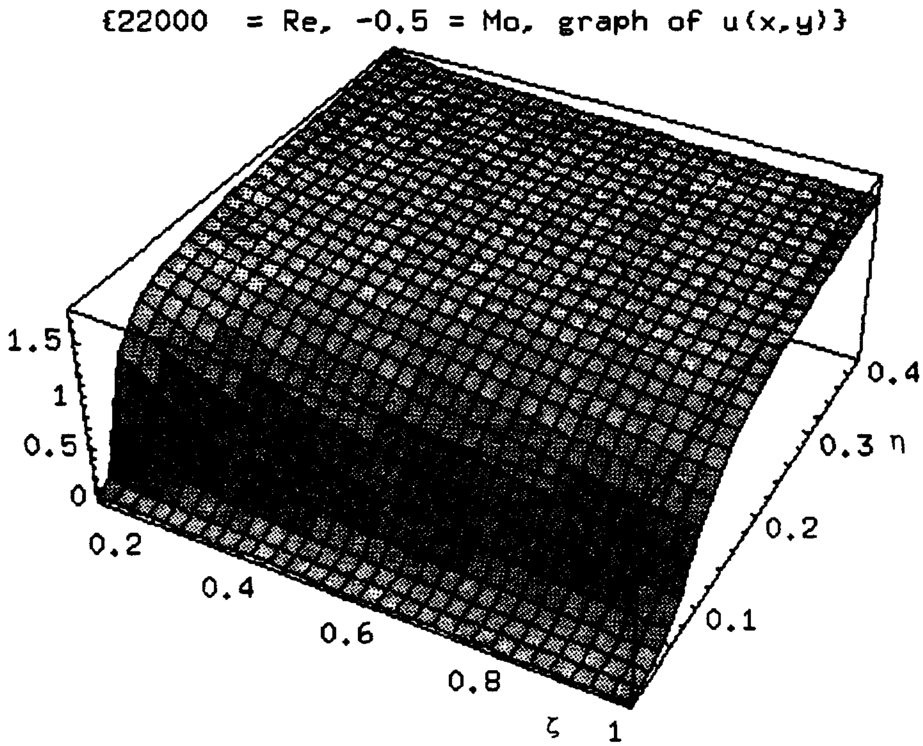


Figure B-16



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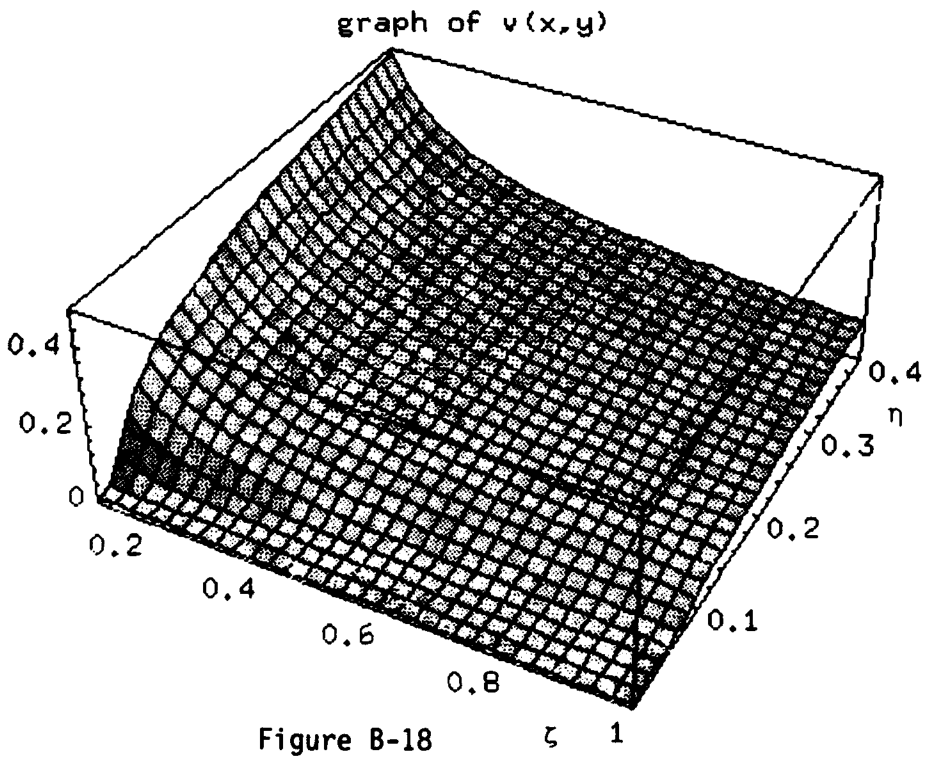

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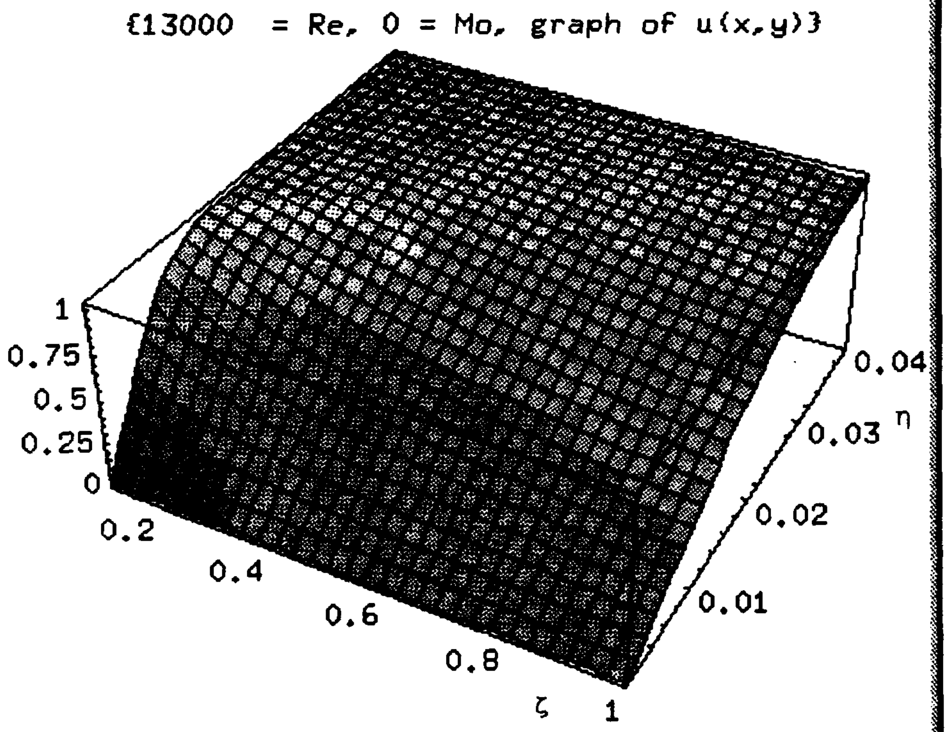

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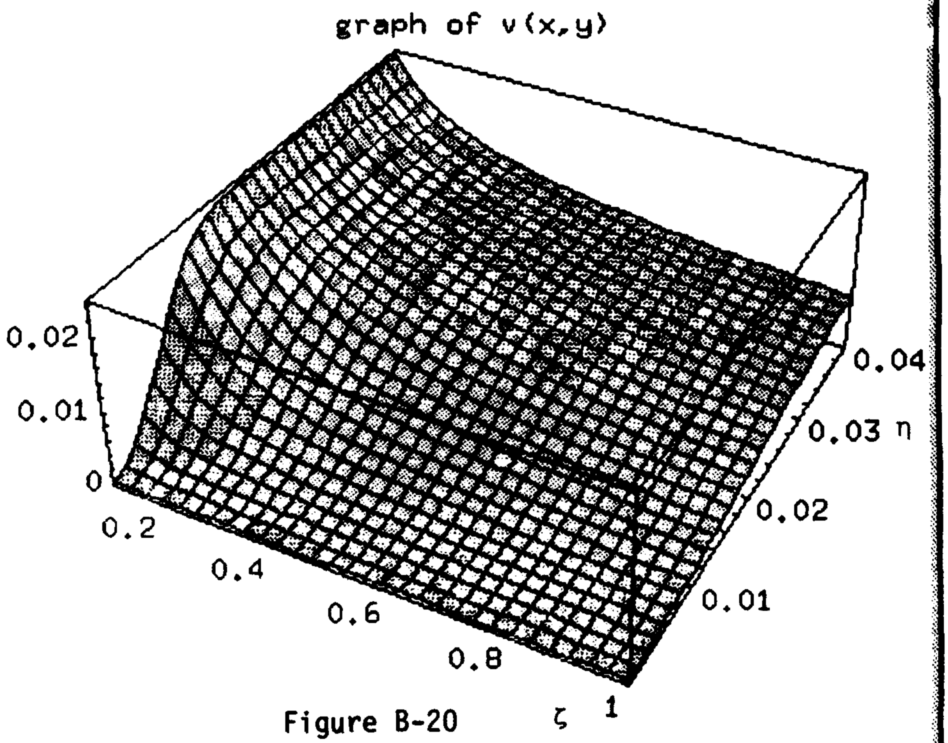

Figure B-20

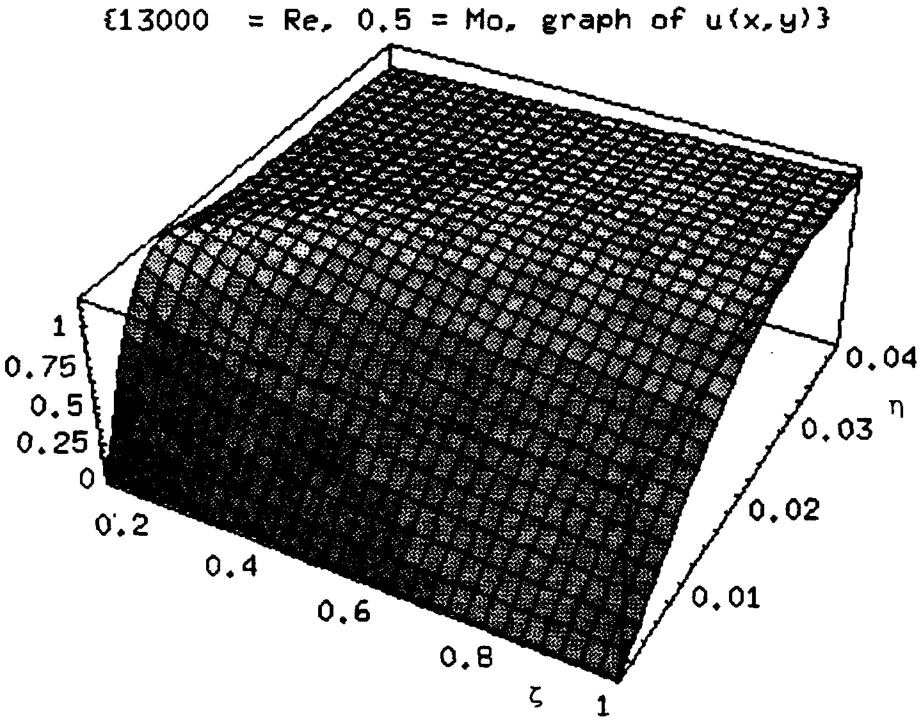

Figure B-21

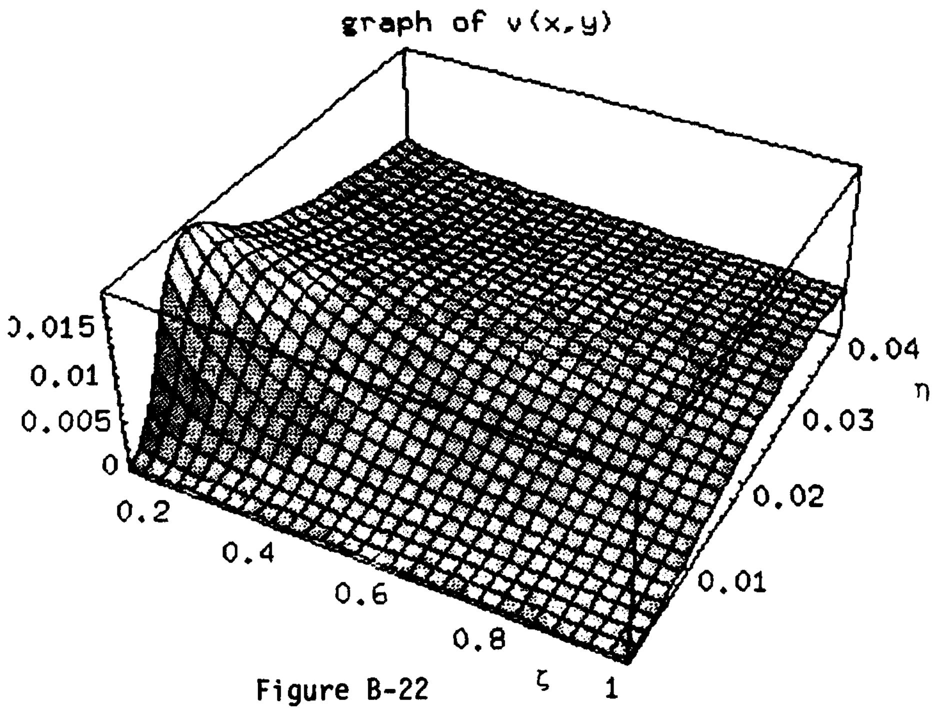

Figure B-22

{13000 = Re, -0.5 = Mo, graph of u(x,y)}

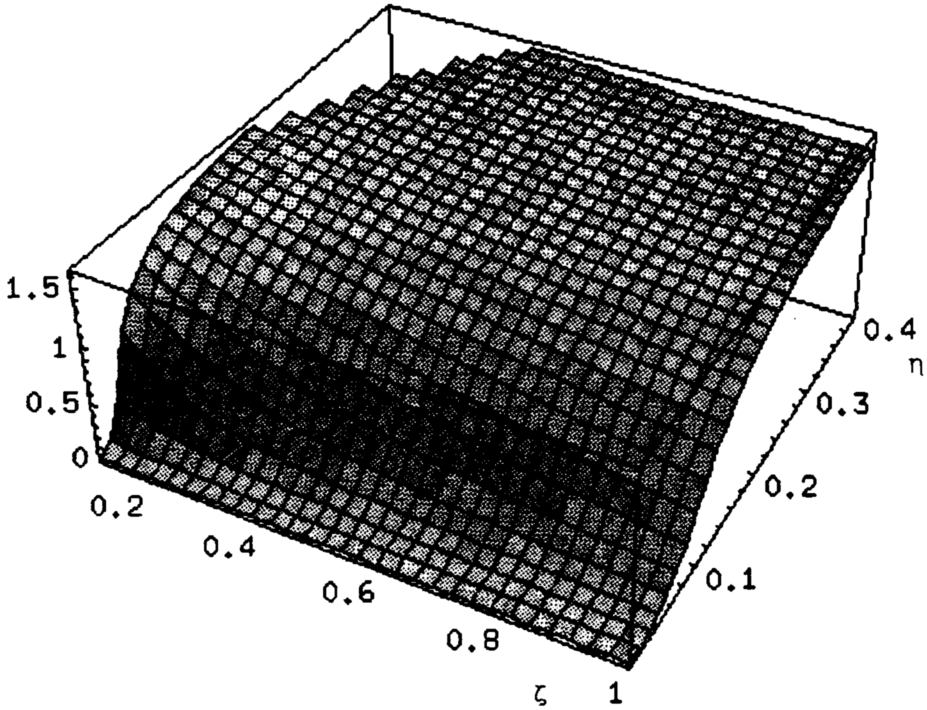

Figure B-23

graph of v(x,y)

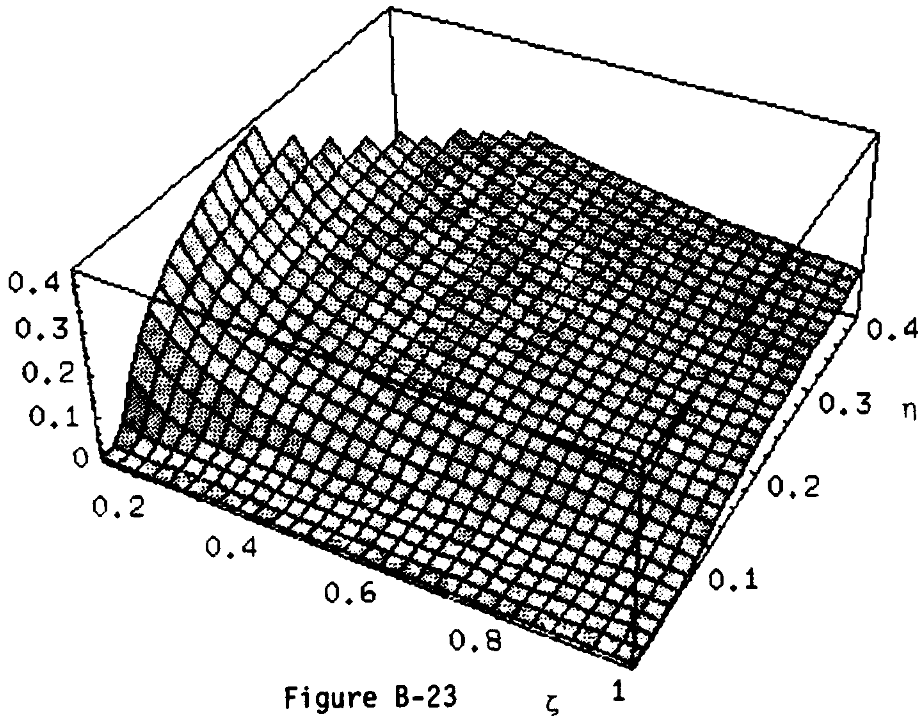

Figure B-23